Math E-21a – Some useful facts

Basic Chain Rule: $\frac{d}{dt}\left[f(x(t),y(t))\right]=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\nabla f\cdot\mathbf{v}$ for a path in $\mathbf{R}^2$;

$$\frac{d}{dt}\left[f(x(t),y(t),z(t))\right]=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}=\nabla f\cdot\mathbf{v}$$ for a path in $\mathbf{R}^3$.

Directional Derivative of a function f in the direction $\mathbf{u}$ (unit vector): $D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}$

Fundamental Theorem of Line Integrals: If V is differentiable and C is a curve from point $\mathbf{x}_0$ to point $\mathbf{x}_1$, then $\int_C \overrightarrow{\nabla V}\cdot\overrightarrow{dr}=V(\mathbf{x}_1)-V(\mathbf{x}_0)$.

Green's Theorem: If $P(x,y)$ and $Q(x,y)$ are differentiable with continuous 1st partial derivatives through a bounded region D in $\mathbf{R}^2$ and if C is the boundary of D oriented in the counterclockwise sense, then $\oint_C P(x,y)\,dx+Q(x,y)\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$.

Curl and Divergence: If $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ is a vector field in $\mathbf{R}^3$ with differentiable component functions, then $\operatorname{curl}\mathbf{F}=\left\langle\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle$ and $\operatorname{div}\mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$.

Divergence Theorem: If the components of the vector field $\mathbf{F}$ are differentiable with continuous 1st partial derivatives through a bounded region B in $\mathbf{R}^3$ and if S is the boundary of B oriented with unit outward normal vector $\mathbf{n}$, then $\oiint_S \mathbf{F}\cdot d\mathbf{S}=\oiint_S(\mathbf{F}\cdot\mathbf{n})\,dS=\iiint_B(\operatorname{div}\mathbf{F})\,dV$.

Stokes' Theorem: If the components of the vector field $\mathbf{F}$ are differentiable with continuous 1st partial derivatives through a surface S in $\mathbf{R}^3$ oriented with unit normal vector $\mathbf{n}$ and if C is the boundary of S oriented counterclockwise relative to $\mathbf{n}$, then $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS=\iint_S(\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}$.

Surface integration "toolkits":

Sphere of radius *R*: $\begin{Bmatrix}x=R\cos\theta\sin\phi\\ y=R\sin\theta\sin\phi\\ z=R\cos\phi\end{Bmatrix}$, $\mathbf{n}=\frac{\langle x,y,z\rangle}{R}$, $dS=R^2\sin\phi\,d\phi\,d\theta$, $x^2+y^2+z^2=R^2$

Cylinder of radius *R*: $\begin{Bmatrix}x=R\cos\theta\\ y=R\sin\theta\\ z=z\end{Bmatrix}$, $\mathbf{n}=\frac{\langle x,y,0\rangle}{R}$, $dS=R\,dz\,d\theta$, $x^2+y^2=R^2$

Graph of *f*(*x*, *y*): $\begin{Bmatrix}x=x\\ y=y\\ z=f(x,y)\end{Bmatrix}$, $\mathbf{n}=\frac{\langle -f_x,-f_y,1\rangle}{\sqrt{1+f_x^{\,2}+f_y^{\,2}}}$, $dS=\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}=\sqrt{1+f_x^{\,2}+f_y^{\,2}}\,dx\,dy$

General parameterized surface: $\mathbf{r}(s,t)=\langle x(s,t),y(s,t),z(s,t)\rangle$, $dS=\left\|\frac{\partial\mathbf{r}}{\partial s}\times\frac{\partial\mathbf{r}}{\partial t}\right\|ds\,dt$, $\overrightarrow{dS}=\left(\frac{\partial\mathbf{r}}{\partial s}\times\frac{\partial\mathbf{r}}{\partial t}\right)ds\,dt$

Geometry formulas: Volume of a ball of radius R: $=\frac{4}{3}\pi R^3$; Surface area of a sphere of radius R: $=4\pi R^2$

Useful identities: $\cos^2\theta=\frac{1+\cos 2\theta}{2}$, $\sin^2\theta=\frac{1-\cos 2\theta}{2}$, $\sin^2\theta+\cos^2\theta=1$