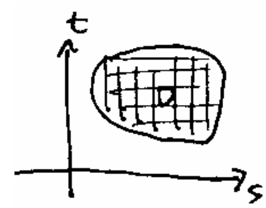
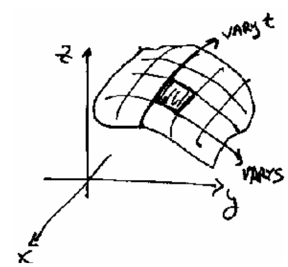
## Integration on Surfaces - Toolkits

It is often necessary to measure the aggregate amount of *something* on a surface. Examples include the total **area** of a surface, total **charge** for a given charge density function, total **population** of people on a planet, or total **flux** (flow) of a vector field through a surface. A general **surface integral** of a function $g(x,y,z)$ over a surface $S$ is denoted by $\iint_S g\,dS$ where, as always, the integral represents the limit of Riemann Sum.

The main tools for calculating such integrals are (a) parameterization of the surface (or, equivalently, finding two coordinates defined on the surface that provide a "mesh" for the surface), and (b) an expression for the "element of surface area" $dS$ defined by the parameterization or coordinates. For flux integrals, it's also handy to have an expression for a unit normal vector $\boldsymbol{n}$ for the surface. Though there is an all-purpose method for calculating surface integrals for any parameterized surface, it is often easier and more geometrically clear to focus on the special cases of cylinders, spheres, and graphs. Each situation has its own **toolkit**.

### General method for any parameterized surface




Parameterization: Suppose $S$ is a surface parameterized by a vector-valued function $\boxed{\mathbf{r}(s,t)=\langle x(s,t),y(s,t),z(s,t)\rangle}$ where the parameters $s$ and $t$ vary over some domain in the parameter space $D$. The only requirement is that the curves in the surface produced by varying one parameter at a time provide a mesh on the surfaces, i.e. these curves should intersect "cleanly" (transversally) and produce a patchwork of small cells on the surface that can be used to build Riemann Sums on the surface.

Surface area element: If we vary $s$ by an amount $\Delta s$, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial s}\Delta s$, and if we also vary $t$ by an amount $\Delta t$, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial t}\Delta t$.

These two displacements will span a "patch" of the surface and the cross-product can be used to determine its approximate area. Specifically,

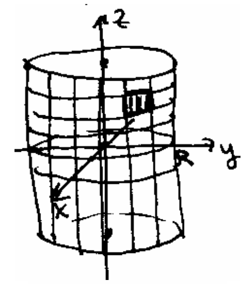
$$\left(\frac{\partial \mathbf{r}}{\partial s}\Delta s\right)\times\left(\frac{\partial \mathbf{r}}{\partial t}\Delta t\right)=\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)\Delta s\Delta t.$$

This is a vector with magnitude approximately equal to the area of a "patch" on the graph surface and direction normal to the surface (actually in the upward normal direction). The magnitude is $\Delta S\cong\left\|\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right\|\Delta s\Delta t$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS=\left\|\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right\|dsdt}$ for a general parameterized surface. We can also make use of the "vector element of surface area" $\boxed{d\mathbf{S}=\mathbf{n}dS=\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)dsdt}$ where $\mathbf{n}$ denotes the unit normal vector to the surface, oriented in a manner consistent with the cross-product.

Unit normal vector: From the above calculation, we also see that a unit normal vector to the surface with orientation consistent with the cross product is $\boxed{\mathbf{n}=\dfrac{\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)}{\left\|\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right\|}}$.

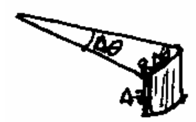
In particular, if $\mathbf{F}=\langle P,Q,R\rangle$ is a vector field defined on the surface, the flux of $\mathbf{F}$ through $S$ can be calculated as $\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\left[\mathbf{F}\cdot\left(\frac{\partial \mathbf{r}}{\partial s}\times\frac{\partial \mathbf{r}}{\partial t}\right)\right]dsdt$. [The integrand is the triple scalar product.]

**Cylinder**




Cartesian equation: $\boxed{x^2+y^2=R^2}$

Parameterization: $\boxed{\begin{array}{l} x=R\cos\theta \\ y=R\sin\theta \\ z=z \end{array}}$. The parameter $\theta$ allows movement around the cylinder with $0\le\theta\le 2\pi$, and the parameter $z$ (which does double-duty as both a coordinate and a parameter) allows movement up and down the cylinder.




Surface area element: If we vary $\theta$ by an amount $\Delta\theta$, we move a distance $R\Delta\theta$ around the cylinder, and if we also move an amount $\Delta z$ to span a "patch" of the surface, the area of this patch will be $\Delta S=(R\Delta z)(\Delta\theta)$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS=Rdzd\theta}$ for a cylinder.

Unit normal vector: We can use gradient methods or observation to see that at any point $(x,y,z)$ on the cylinder, the outward unit normal vector to the surface is $\boxed{\mathbf{n}=\frac{\langle x,y,0\rangle}{R}=\langle\cos\theta,\sin\theta,0\rangle}$.

This method can be modified as necessary for cylinders around the $x$-axis or $y$-axis.

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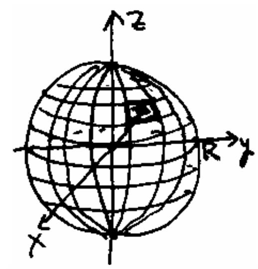
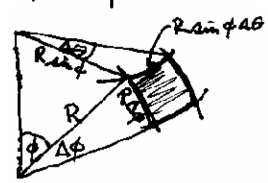
**Sphere**




Cartesian equation: $\boxed{x^2+y^2+z^2=R^2}$

Parameterization: $\boxed{\begin{array}{l} x=R\cos\theta\sin\phi \\ y=R\sin\theta\sin\phi \\ z=R\cos\phi \end{array}}$. The parameter $\theta$ (longitude) allows movement around the sphere with $0\le\theta\le 2\pi$, and the parameter $\phi$ (the inclination, related to latitude) allows movement up and down the sphere with $0\le\phi\le\pi$.




Surface area element: If we vary $\phi$ by an amount $\Delta\phi$, we move a distance $R\Delta\phi$ along a longitude, and if we also move an amount $\Delta\theta$ along a latitude (at a radius from the $z$-axis of $r=R\sin\phi$), we will move a distance $R\sin\phi\,\Delta\theta$ along a latitude. Together, these will span a "patch" of the surface with area approximately $\Delta S\cong(R\sin\phi\,\Delta\theta)(R\Delta\phi)=R^2\sin\phi\,\Delta\theta\,\Delta\phi$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS=R^2\sin\phi\,d\phi d\theta}$ for a sphere.

Unit normal vector: We can use gradient methods or observation to see that at any point $(x,y,z)$ on the sphere, the outward unit normal vector to the surface is $\boxed{\mathbf{n}=\frac{\langle x,y,z\rangle}{R}=\langle\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi\rangle}$.

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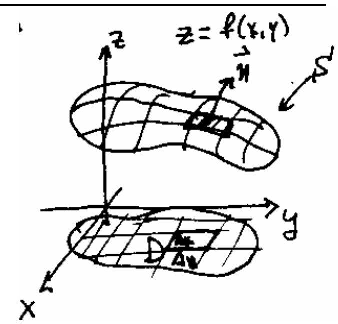
**Graph** $z=f(x,y)$




Cartesian equation: $\boxed{z=f(x,y)}$

Parameterization: $\boxed{\begin{array}{l} x=x \\ y=y \\ z=f(x,y) \end{array}}$ or $\boxed{\mathbf{r}(x,y)=\langle x,y,f(x,y)\rangle}$. The variables $x$ and $y$ here do double-duty as both parameters and coordinates. They vary in the $xy$-plane over the domain of the function that describes this graph surface.

Surface area element: If we vary $x$ by an amount $\Delta x$, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial\mathbf{r}}{\partial x}\Delta x=\langle 1,0,f_x\rangle\Delta x$, and if we also vary $y$ by an amount $\Delta y$, we move an approximate vector displacement along a cross-section of the graph of $\frac{\partial \mathbf{r}}{\partial y}\Delta y = \langle 0,1,f_y\rangle \Delta y$. Together, these two displacements will span a "patch" of the surface and we can use the cross-product to determine its approximate area. Specifically,

$$\left(\frac{\partial \mathbf{r}}{\partial x}\Delta x\right)\times\left(\frac{\partial \mathbf{r}}{\partial y}\Delta y\right)=\left(\frac{\partial \mathbf{r}}{\partial x}\times\frac{\partial \mathbf{r}}{\partial y}\right)\Delta x\Delta y=\langle 1,0,f_x\rangle\times\langle 0,1,f_y\rangle\Delta x\Delta y=\langle -f_x,-f_y,1\rangle\Delta x\Delta y\,.$$

This is a vector with magnitude approximately equal to the area of a "patch" on the graph surface and direction normal to the surface (actually in the upward normal direction). The magnitude is $\Delta S \cong \sqrt{1+f_x^{\,2}+f_y^{\,2}}\,\Delta x\Delta y$. Within a Riemann Sum expression as the limit of the mesh tends to zero, this yields the surface area element $\boxed{dS \cong \sqrt{1+f_x^{\,2}+f_y^{\,2}}\,dxdy}$ for a graph surface. We can also make use of the "vector element of surface area" $\boxed{d\mathbf{S}=\mathbf{n}dS=\langle -f_x,-f_y,1\rangle\,dxdy}$ where **n** denotes the (upward) unit normal vector to the graph surface.

<u>Unit normal vector</u>: From the above calculation, we see that a unit (upward) normal vector to a graph surface is $\boxed{\mathbf{n}=\frac{\langle -f_x,-f_y,1\rangle}{\sqrt{1+f_x^{\,2}+f_y^{\,2}}}}$. An alternative geometric argument also shows that $\boxed{dS=\frac{dxdy}{\mathbf{n}\cdot\mathbf{k}}=\sqrt{1+f_x^{\,2}+f_y^{\,2}}\,dxdy}$.

<u>Note</u>: [This can be easily adapted to the case where $x=f(y,z)$ with $\boxed{dS=\frac{dydz}{\mathbf{n}\cdot\mathbf{i}}=\sqrt{1+f_y^{\,2}+f_z^{\,2}}\,dydz}$ or where $y=f(x,z)$ with $\boxed{dS=\frac{dxdz}{\mathbf{n}\cdot\mathbf{j}}=\sqrt{1+f_x^{\,2}+f_z^{\,2}}\,dxdz}$.]

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**Examples**:

(1) <u>Surface area</u> of a sphere $S$ of radius $R$:

$\text{Area}(S)=\iint_S dS=\int_0^{2\pi}\int_0^{\pi}R^2\sin\phi\,d\phi d\theta$. The inner integral gives $-R^2\cos\phi\Big|_{\phi=0}^{\phi=\pi}=-R^2(-1-1)=2R^2$, and the outer integral gives $(2R^2)(2\pi)=4\pi R^2$.

(2) The <u>flux of the vector field</u> $\mathbf{F}=\langle 3x,-y,z^2\rangle$ outward through a sphere of radius 2 centered at the origin:

$\text{Flux}=\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_S \mathbf{F}\cdot\mathbf{n}dS$. We use the unit outward normal vector $\mathbf{n}=\frac{\langle x,y,z\rangle}{2}$, so $\mathbf{F}\cdot\mathbf{n}=\langle 3x,-y,z^2\rangle\cdot\frac{\langle x,y,z\rangle}{2}=\frac{3x^2-y^2+z^3}{2}$. Therefore, $\text{Flux}=\iint_S\left(\frac{3x^2-y^2+z^3}{2}\right)dS$. If we substitute the parameterization $\left\{\begin{array}{l}x=2\cos\theta\sin\phi\\ y=2\sin\theta\sin\phi\\ z=2\cos\phi\end{array}\right\}$ and the area element $dS=4\sin\phi\,d\phi d\theta$, we get

$\text{Flux}=\int_0^{2\pi}\int_0^{\pi}(24\cos^2\theta\sin^3\phi-8\sin^2\theta\sin^3\phi+16\cos^3\phi\sin\phi)\,d\phi d\theta$.

This may not be the simplest integral, but it's quite doable. The <u>inner integral</u> gives

$$\int_0^{\pi}(24\cos^2\theta(1-\cos^2\phi)\sin\phi-8\sin^2\theta(1-\cos^2\phi)\sin\phi+16\cos^3\phi\sin\phi)\,d\phi$$

$$=[24\cos^2\theta-8\sin^2\theta]\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi}-4\left[\cos^4\phi\right]_0^{\pi}=[24\cos^2\theta-8\sin^2\theta]\left[\tfrac{4}{3}\right]-4[0]$$

$$=\tfrac{4}{3}\left[24\left(\frac{1+\cos2\theta}{2}\right)-8\left(\frac{1-\cos2\theta}{2}\right)\right]=\tfrac{4}{3}(8+16\cos2\theta)$$

The outer integral is then $\frac{4}{3}\int_0^{2\pi}(8+16\cos 2\theta)\,d\theta = \frac{32}{3}\cdot 2\pi = \frac{64\pi}{3}$.

Note: This integral can also be (more simply) done using the **Divergence Theorem**. We calculate $\operatorname{div}\mathrm{F} = 3-1+2z = 2+2z$, and

$$\text{Flux} = \oiint_{S=\mathrm{Bnd}(B)} \mathbf{F}\cdot d\mathbf{S} = \iiint_B (\operatorname{div}\mathbf{F})\,dV = \iiint_B (2+2z)\,dV = \iiint_B 2\,dV = 2\cdot\mathrm{Vol}(B) = 2\cdot(\tfrac{4}{3}\pi\cdot 2^3) = \frac{64\pi}{3}.$$

(3) Surface area of the paraboloid $z = x^2+y^2$ lying over the disk $x^2+y^2\le 4$ in the $xy$-plane.

There are several good approaches. If were to use $(x,y)$ as parameters, we might describe the paraboloid parametrically by $\left\{\begin{matrix} x = x \\ y = y \\ z = x^2+y^2 \end{matrix}\right\}$ or $\mathbf{r}(x,y) = \langle x, y, x^2+y^2\rangle$. The methods described above (with $z = f(x,y)$) give us that $dS = \sqrt{1+f_x^{\,2}+f_y^{\,2}}\,dxdy = \sqrt{1+(2x)^2+(2y)^2}\,dxdy = \sqrt{1+4(x^2+y^2)}\,dxdy$, so the surface area would be $\iint_D \sqrt{1+4(x^2+y^2)}\,dxdy$. The sensible thing is to change to polar coordinates to calculate this integral over the disk. This gives $\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,rdrd\theta = \cdots = \boxed{\frac{\pi}{6}(17\sqrt{17}-1)}$.

We could also have begun by using $(r,\theta)$ as parameters. This would give $\left\{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{matrix}\right\}$ or $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$. We calculate $\frac{\partial\mathbf{r}}{\partial r} = \langle\cos\theta,\sin\theta,2r\rangle$ and $\frac{\partial\mathbf{r}}{\partial\theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$, so

$$d\mathbf{S} = \frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle = r\langle -2r\cos\theta, -2r\sin\theta, 1\rangle \text{ and}$$

$$dS = \left\|\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right\| drd\theta = r\left\|\langle -2r\cos\theta,-2r\sin\theta,1\rangle\right\| drd\theta = r\sqrt{4r^2(\cos^2\theta+\sin^2\theta)+1}\,drd\theta = r\sqrt{1+4r^2}\,drd\theta \text{, so}$$

we again get $\mathrm{Area}(S) = \iint_S dS = \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,rdrd\theta = \cdots = \boxed{\frac{\pi}{6}(17\sqrt{17}-1)}$.