Basic Facts - Dot Products and Cross Products

In addition to the most basic operations of scaling and vector addition (both done component-wise), the measurement of lengths and angles are facilitated by the dot product of vectors (also known as the inner product). The dot product can be defined in $\mathbf{R}^n$ for any n which allows for the definition of orthogonality in any dimension. In addition, the cross product (defined only in $\mathbf{R}^3$) allows us construct a vector orthogonal to any pair of vectors and also to measure areas of parallelograms.

Definition: The **dot product** of two <u>vectors</u> **u**, **v** in $\mathbf{R}^n$ is a <u>scalar</u> defined as follows:

$$\mathbf{u}\cdot\mathbf{v}=\langle u_1,u_2,\cdots,u_n\rangle\cdot\langle v_1,v_2,\cdots,v_n\rangle=u_1v_1+u_2v_2+\cdots+u_nv_n$$

There are some easy-to-verify algebraic properties of the dot product that follow from this definition:

Algebraic Properties of the Dot Product: Suppose **u**, **v**, and **w** are vectors in $\mathbf{R}^n$ and that t is any scalar.

1) $\mathbf{v}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{v}$ (symmetry, dot product is commutative)

2) $\left\{\begin{matrix}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}\\(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}\end{matrix}\right\}$ (left and right distributive laws)

3) $(t\,\mathbf{u})\cdot\mathbf{v}=t(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(t\,\mathbf{v})$ (how the dot product behaves relative to scaling of vectors)

4) $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2\geq 0$ for all **u** (and $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2=0$ only for $\mathbf{u}=\mathbf{0}$)

Using these algebraic properties and the Law of Cosines (a corollary of the Pythagorean Theorem) we were able to derive the following important property of the dot product:
If **u**, **v** in $\mathbf{R}^n$ are two vectors emanating out from a common vertex to form an angle θ, and if $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are their respective lengths, then $\boxed{\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta}$. The great importance of this relation is that connects the algebraically-defined dot product to the geometric measurements of lengths and angles.

We immediately get the following corollary using some basic trigonometric facts: If **u**, **v** in $\mathbf{R}^n$ are <u>nonzero</u> vectors emanating from a common vertex to form an angle θ, then

$\mathbf{u}\cdot\mathbf{v}>0$ if and only if the angle θ is acute

$\mathbf{u}\cdot\mathbf{v}<0$ if and only if the angle θ is obtuse

$\mathbf{u}\cdot\mathbf{v}=0$ if and only if the angle θ is a right angle, i.e. $\mathbf{u}\perp\mathbf{v}$

We can also use the relation $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ (and a sketch) to define the **scalar projection** of a vector **v** in the direction of another vector **u** (also called the **component** of **v** in the direction of **u**) as $\boxed{\mathbf{v}\cdot\dfrac{\mathbf{u}}{\|\mathbf{u}\|}}$. This is perhaps best remembered by noting that to find the component of a vector **v** in any given direction, you "dot **v** with a <u>unit vector</u> in that direction". We can then use this fact to define the vector projection of **v** in the direction of **u** by construction it as $\boxed{\mathrm{Proj}_{\mathbf{u}}\mathbf{v}=\left(\mathbf{v}\cdot\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right)\dfrac{\mathbf{u}}{\|\mathbf{u}\|}=\left(\dfrac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right)\mathbf{u}}$. This can be useful for expressing a vector as the sum of a "tangential component" vector and a "normal component" vector, especially in geometry and physics.

The fact that the orthogonality of vectors can be characterized algebraically by their dot product being zero allowed us to derive that the equation of a plane with normal vector **n** and passing through a point with position vector $\mathbf{x}_0$ must be of the form $\boxed{\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)=0}$ where $\mathbf{x}$ represents the position vector of any other point on the plane. In $\mathbf{R}^3$, if we express this in components with $\mathbf{n}=\langle A,B,C\rangle$, $\mathbf{x}_0=\langle x_0,y_0,z_0\rangle$, and $\mathbf{x}=\langle x,y,z\rangle$, this becomes $\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$ or $\boxed{Ax+By+Cz=D}$ where D is the constant obtained after multiply out and transposing constants to the right-hand-side. In problems, we often jump to this form once we know the normal vector and determine D by plugging in the coordinates of the given point.

It is sometimes the case that we need to find the equation of a plane given not a normal vector and a single point, but rather three non-colinear points. In this case, we can take points pairwise to produce vectors parallel to the plane and may desire to use these to find a vector orthogonal to the plane. A convenient way to do this is via the **cross product** (defined only in $\mathbf{R}^3$). Given two vectors $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ in $\mathbf{R}^3$, we can use the orthogonality requirement to show that the following **cross product** will be orthogonal to both vectors:

$$\mathbf{u}\times\mathbf{v}=\langle u_1,u_2,u_3\rangle\times\langle v_1,v_2,v_3\rangle=\langle u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1\rangle$$

There are several different ways to express this using the definition of a 2×2 determinant, namely $\det\begin{bmatrix} a & b \\ c & d\end{bmatrix}=\begin{vmatrix} a & b \\ c & d\end{vmatrix}=ad-bc$. Examining the above expression we see that:

$$\mathbf{u}\times\mathbf{v}=\left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3\end{vmatrix},\begin{vmatrix} u_3 & u_1 \\ v_3 & v_1\end{vmatrix},\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix}\right\rangle=\left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3\end{vmatrix},-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3\end{vmatrix},\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix}\right\rangle$$

Note the sign switch in the middle component. This is done so that you can conveniently perform the calculation by creating a 2×3 array from the given two vectors $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{bmatrix}$ and then respectively covering the 1st, 2nd, and 3rd columns and calculating the determinant of the resulting 2×2 determinants (with appropriate sign switch of the middle component. For example, if $\mathbf{u}=\langle 1,3,6\rangle$ and $\mathbf{v}=\langle -2,5,4\rangle$, we would get the array $\begin{bmatrix} 1 & 3 & 6 \\ -2 & 5 & 4\end{bmatrix}$ and use the procedure to calculate $\mathbf{u}\times\mathbf{v}=\langle 12-30,-(4+12),5+6\rangle=\langle -18,-16,11\rangle$. A quick check using the dot product shows that this is orthogonal to both **u** and **v**.

Some people prefer to express this procedure using $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ notation by formally calculating the 3×3 determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix}=\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix}\mathbf{k}$.

Using only this algebraic definition for the cross product, we can derive the following properties:

Algebraic Properties of the Cross Product: Suppose **u**, **v**, and **w** are vectors in $\mathbf{R}^3$ and that t is any scalar.

1) $\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}$ (anticommutative) [Corollary: $\mathbf{u}\times\mathbf{u}=\mathbf{0}$ for any vector $\mathbf{u}$]
2) $\left\{\begin{matrix}\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w} \\ (\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}\end{matrix}\right\}$ (left and right distributive laws)
3) $(t\,\mathbf{u})\times\mathbf{v}=t(\mathbf{u}\times\mathbf{v})=\mathbf{u}\times(t\,\mathbf{v})$ (how the dot product behaves relative to scaling of vectors)
4) $\mathbf{u}\times\mathbf{0}=\mathbf{0}$
5) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ (triple scalar product)
6) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (triple vector product)

All of the above algebraic properties of the cross product except for the last one are straightforward. You can prove the last one by noting that the first component would be:

$$\begin{vmatrix} u_2 & u_3 \\ v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} = u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) = u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3$$

$$= u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3 + u_1v_1w_1 - u_1v_1w_1 = (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1$$

$$= (\mathbf{u}\cdot\mathbf{w})v_1 - (\mathbf{u}\cdot\mathbf{v})w_1$$

Similarly, we can show that the 2nd and 3rd components are $(\mathbf{u}\cdot\mathbf{w})v_2 - (\mathbf{u}\cdot\mathbf{v})w_2$ and $(\mathbf{u}\cdot\mathbf{w})v_3 - (\mathbf{u}\cdot\mathbf{v})w_3$.

Together these give that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$. Physicists (and others) often refer to this property as the "BAC-CAB Rule" and express it as $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

We can independent define the cross product in purely geometric terms.

Geometric definition of the cross product: Suppose **u** and **v** are vectors in $\mathbf{R}^3$. Then the cross product $\mathbf{u}\times\mathbf{v}$ is the unique vector in $\mathbf{R}^3$ such that:

(1) $\mathbf{u}\times\mathbf{v}$ is orthogonal to both u and v;
(2) the magnitude of the cross product $\|\mathbf{u}\times\mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**;
(3) $\mathbf{u}\times\mathbf{v}$ is oriented according to the Right-Hand Rule (as explained in class and elsewhere).

It is true that these three properties uniquely determine the cross product, and we can also easily derive the previous algebraic definition from these requirements. We can also derive these geometric properties from the algebraic definition using the previously stated algebraic properties. Specifically:

(1) $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = (\mathbf{u}\times\mathbf{u})\cdot\mathbf{v} = \mathbf{0}\cdot\mathbf{v} = 0$, so $\mathbf{u}\times\mathbf{v}$ is orthogonal to the vector **u**;
$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$, so $\mathbf{u}\times\mathbf{v}$ is orthogonal to the vector **v**.

(2) If we consider the parallelogram determined by **u** and **v** and let θ be the angle between these vectors (drawing a picture is advisable), then the area of the parallelogram will be given by
$(\text{length of base})(\perp\text{ height}) = \|\mathbf{u}\|\|\mathbf{v}\||\sin\theta|$. Squaring both sides gives

$(\text{Area})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$

On the other hand, $\|\mathbf{u}\times\mathbf{v}\|^2 = (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot[\mathbf{v}\times(\mathbf{u}\times\mathbf{v})] = \mathbf{u}\cdot[(\mathbf{v}\cdot\mathbf{v})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{v}] = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$

Therefore $(\text{Area})^2 = \|\mathbf{u}\times\mathbf{v}\|^2$, so $\text{Area} = \|\mathbf{u}\times\mathbf{v}\|$.

(3) You can easily calculate using the algebraic definition that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ which satisfies the Right-Hand Rule. Then argue using a continuity argument that if this is true for these two vectors than by continuously varying these vectors in $\mathbf{R}^3$ to align them with the given two vectors, the right-hand rule must be preserved.

Volume and the Triple Scalar Product: We showed that, up to sign, the volume of the parallelepiped determined by the three vectors **u**, **v**, and **w** in $\mathbf{R}^3$ is given by the triple scalar product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

That is, the volume is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ or $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$.