Orthogonal Curvilinear Coordinates: Div, Grad, Curl, and the Laplacian

The most common way that the gradient of a function, the divergence of a vector field, and the curl of a vector field are presented is entirely algebraic with barely any indication of what these mean. Furthermore, the presentation is almost always in terms of the standard Euclidean coordinates (x, y, z) for $\mathbf{R}^3$.

For a function $f(x, y, z)$ and a vector field $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$, we have:

$$\overrightarrow{\nabla f} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \qquad \operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

It is often the case that a vector field in $\mathbf{R}^3$ is most easily understood in coordinates other than standard Euclidean coordinates, e.g. a radial vector field associated with an inverse square law such as **gravitational force** of a point mass, or the **electric field** of a point charge.

Suppose we have coordinates (u_1, u_2, u_3) and that we can write the standard (x, y, z) coordinates in terms of the (u_1, u_2, u_3) coordinates, i.e. $(x, y, z) = \mathbf{r}(u_1, u_2, u_3) = \big(x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)\big)$. By varying each coordinate independently, we can produce tangent vectors in each direction: $\frac{\partial \mathbf{r}}{\partial u_1}$, $\frac{\partial \mathbf{r}}{\partial u_2}$, and $\frac{\partial \mathbf{r}}{\partial u_3}$. We can then normalize each of these tangent vectors to produce unit vectors in these three independent directions. These are often denoted (especially in physics) by placing a "hat" on the respective variable, i.e. unit vectors $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$. If these vectors are mutually perpendicular (orthogonal) at every point, we refer to the coordinates (u_1, u_2, u_3) as orthogonal curvilinear coordinates.

For example, Euclidean (Cartesian) coordinates are curvilinear with $\hat{u}_1 = \mathbf{i}$, $\hat{u}_2 = \mathbf{j}$, and $\hat{u}_3 = \mathbf{k}$.

In spherical coordinates (ρ, ϕ, θ), we have $(x, y, z) = \mathbf{r}(\rho, \phi, \theta) = \big(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\big)$ and the unit direction vectors are $\hat{\rho}$ (unit vector radially outward), $\hat{\phi}$ (unit southward vector tangent to great circles), and $\hat{\theta}$ (unit eastward vector tangent to latitudes). These are easily calculated to be:

$$\begin{aligned}
\hat{\rho} &= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle = (\cos\theta\sin\phi)\,\mathbf{i} + (\sin\theta\sin\phi)\,\mathbf{j} + (\cos\phi)\,\mathbf{k} \\
\hat{\phi} &= \langle\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle = (\cos\theta\cos\phi)\,\mathbf{i} + (\sin\theta\cos\phi)\,\mathbf{j} - (\sin\phi)\,\mathbf{k} \\
\hat{\theta} &= \langle-\sin\theta, \cos\theta, 0\rangle = -(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}
\end{aligned}$$

We can, of course, do the same in $\mathbf{R}^2$, but the divergence and curl of a vector field have a more natural meaning in $\mathbf{R}^3$, so we'll focus on that case. The formulation of the 2D case is left as an exercise.

If we define the "scale factors" as $h_1 = \left\|\frac{\partial \mathbf{r}}{\partial u_1}\right\|$, $h_2 = \left\|\frac{\partial \mathbf{r}}{\partial u_2}\right\|$, and $h_3 = \left\|\frac{\partial \mathbf{r}}{\partial u_3}\right\|$, then an incremental change in position $d\mathbf{r}$ will be expressible as $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3 = (h_1\,du_1)\,\hat{u}_1 + (h_2\,du_2)\,\hat{u}_2 + (h_3\,du_3)\,\hat{u}_3$; the element of arc length ds will satisfy $ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$; and the volume element will be expressible as $dV = h_1 h_2 h_3\, du_1 du_2 du_3$.

For example, in spherical coordinates we have $h_1 = \left\| \frac{\partial \mathbf{r}}{\partial \rho} \right\| = 1$, $h_2 = \left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\| = \rho$, and $h_3 = \left\| \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \rho \sin\phi$. We then have the vector displacement $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial \theta} d\theta = (d\rho)\,\hat{\rho} + (\rho d\phi)\,\hat{\phi} + (\rho \sin\phi\, d\theta)\,\hat{\theta}$. The increment of arclength ds satisfies $ds^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi\, d\theta^2$ and the volume element is $dV = \rho^2 \sin\phi\, d\rho d\phi d\theta$.

For any function $f(u_1,u_2,u_3)$ and a vector field $\mathbf{F}(u_1,u_2,u_3) = A_1(u_1,u_2,u_3)\,\hat{u}_1 + A_2(u_1,u_2,u_3)\hat{u}_2 + A_3(u_1,u_2,u_3)\,u_3$ (note that A_1, A_2, A_3 are the component functions of **F** relative to the orthogonal curvilinear coordinates), we define the gradient vector, divergence, and curl as follows:

$$\text{grad } f = \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\right)\hat{u}_1 + \left(\frac{1}{h_2}\frac{\partial f}{\partial u_2}\right)\hat{u}_2 + \left(\frac{1}{h_3}\frac{\partial f}{\partial u_3}\right)\hat{u}_3$$

$$\text{div } \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

$$\text{curl } \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 A_3) - \frac{\partial}{\partial u_3}(h_2 A_2)\right]\hat{u}_1 + \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(h_1 A_1) - \frac{\partial}{\partial u_1}(h_3 A_3)\right]\hat{u}_2 + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 A_2) - \frac{\partial}{\partial u_2}(h_1 A_1)\right]\hat{u}_3$$

Exercise 1. Show that these general expressions yield the usual formulas for the gradient of a function, the divergence of a vector field, and the curl of a vector field in Euclidean (x, y, z) coordinates.

Exercise 2. Calculate the expressions for the gradient of a function, the divergence of a vector field, and the curl of a vector field in cylindrical (r, θ, z) coordinates.

Exercise 3. Calculate the expressions for the gradient of a function, the divergence of a vector field, and the curl of a vector field in spherical (ρ, ϕ, θ) coordinates.

Exercise 4. Use the above result to calculate the divergence of a vector field in $\mathbf{R}^3$ governed by an inverse square law, i.e. a vector field of the form $\mathbf{F} = \frac{k}{\rho^2}\hat{\rho}$.

One other important function encountered in the study of waves and heat flow (as well as in quantum mechanics) is the Laplacian of a function *f*. It is often denoted by $\nabla^2 f$ because it is defined (in Cartesian coordinates) by $\nabla^2 f = \vec{\nabla} \cdot \overrightarrow{\nabla f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

As in the case of the gradient, divergence, and curl, this algebraic definition tells us very little about what the Laplacian measures, and the definition is very specific to Euclidean (x, y, z) coordinates.

The more general definition of the Laplacian in orthogonal curvilinear coordinates (u_1, u_2, u_3) is as follows:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

Exercise 5: Show that this expression yields the usual formula for the Laplacian of a function in Euclidean (x, y, z) coordinates.

Exercise 6: Calculate the expression for the Laplacian of a function in cylindrical (r, θ, z) coordinates.

Exercise 7: Calculate the expression for the Laplacian of a function in spherical (ρ, ϕ, θ) coordinates.

Exercise 8. (a) Formulate an expression for the two-dimension divergence of a vector field in $\mathbf{R}^2$ in orthogonal curvilinear coordinates (u_1, u_2). [In Cartesian coordinates, if $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$, the 2D-divergence is defined as $\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.]

(b) Use this general expression to find a formula for the 2D-divergence of a vector field given in polar coordinates as $\mathbf{F}(r, \theta) = L(r, \theta)\,\hat{r} + M(r, \theta)\,\hat{\theta}$.

(c) Use the above result to calculate the 2D-divergence of a vector field governed by an inverse square law in $\mathbf{R}^2$, i.e. a vector field of the form $\mathbf{F} = \frac{k}{r^2}\hat{r}$.

Exercise 9. (a) Formulate an expression for the two-dimension curl of a vector field in $\mathbf{R}^2$ in orthogonal curvilinear coordinates (u_1, u_2). [In Cartesian coordinates, if $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$, the 2D-curl is defined as $\operatorname{curl} \mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.]

(b) Use this general expression to find a formula for the 2D-curl of a vector field given in polar coordinates as $\mathbf{F}(r, \theta) = L(r, \theta)\,\hat{r} + M(r, \theta)\,\hat{\theta}$.

(c) Use the above result to calculate the 2D-curl of a vector field governed by an inverse square law in $\mathbf{R}^2$, i.e. a vector field of the form $\mathbf{F} = \frac{k}{r^2}\hat{r}$.

(d) More generally, calculate the 2D-divergence and the 2D-curl of a radial vector field of the form $\mathbf{F} = \frac{k}{r^p}\hat{r}$ for any *p*. For what value(s) of *p*, if any, will the divergence be identically zero at all points?