Math E-21a – HW #11 supplementary notes
Conservative Vector Fields and the Fundamental Theorem of Line Integrals

For this week's assignment, there are a number of problems (section 13.3) that involve conservative vector fields, potential functions, and the Fundamental Theorem of Line Integrals. Though these problems are all relatively easy to do, we may not get to all of these topics in this week's lecture (except for a reference to the fact that **conservative vector fields** and **gradient vector fields** are one and the same).

The ideas and necessary theorems are all relatively easy to state and apply. Though I'm sure you can all read the necessary information in whichever text you are using, some supplementary notes are provided below to highlight the essential ideas.

Work Integrals

It's essential to keep in mind that when calculating a path integral of the form $\int_C \mathbf{F}\cdot d\mathbf{r}$ (known as a "line integral" even though it has nothing to do with lines), the value of this integral from one point to another will generally depend on which path is taken. The idea is derived from physics and basically means that the work (energy) done by a vector field (representing a force) as you travel from one point to another may well depend on the path followed. **Only in the case of a conservative vector field** (the name comes from conservation of energy) **will the work be independent of the path**. This, in fact, is what defines a conservative vector field.

For a vector field in $\mathbf{R}^2$ given as $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle = P(x,y)\,\mathbf{i}+Q(x,y)\,\mathbf{j}$ and a parameterized curve given as $\mathbf{r}(t)=\langle x(t),y(t)\rangle = x(t)\,\mathbf{i}+y(t)\,\mathbf{j}$, we derived a string of equivalent expressions for the work integral associated with this vector field and parameterized curve. The following facts facilitated this string of equivalent expressions:

$$\mathbf{v}(t)=\frac{d\mathbf{r}}{dt}=\mathbf{r}'(t),\quad d\mathbf{r}=\mathbf{v}(t)dt,\quad T=\frac{\mathbf{v}}{\|\mathbf{v}\|},\quad F_T=\mathbf{F}\cdot\mathbf{T},\quad \frac{ds}{dt}=\|\mathbf{v}(t)\|,\quad ds=\|\mathbf{v}\|dt$$

With these, we have (assuming the parameter t varies between a and b):

$$\text{Work}=\int_C F_T\,ds=\int_C \mathbf{F}\cdot\mathbf{T}\,ds=\int_a^b \mathbf{F}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|}\|\mathbf{v}\|dt=\int_a^b \mathbf{F}\cdot\mathbf{v}\,dt=\int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt=\int_C \mathbf{F}\cdot d\mathbf{r}$$

Furthermore, if we formally write $d\mathbf{r}=\langle dx,dy\rangle$ and $\mathbf{F}=\langle P,Q\rangle$, this can also be written as $\int_C Pdx+Qdy$.
In the case where a vector field in $\mathbf{R}^3$ is given as $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle = P\,\mathbf{i}+Q\,\mathbf{j}+R\,\mathbf{k}$ and the "little displacement vector" is $d\mathbf{r}=\langle dx,dy,dz\rangle$, the work integral becomes $\int_C Pdx+Qdy+Rdz$.

We calculated a work integral like this in class following two different paths and found two different values for the work. This raises the question of when might the work integral be **independent of path**?
It's easy to answer this question one way (gradient implies conservative):

> **Fundamental Theorem of Line Integrals**: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a\le t\le b$ and let $\mathbf{F}=\overrightarrow{\nabla V}$ where $V(x,y)$ (or $V(x,y,z)$) is a differentiable function of two (or three) variables whose gradient $\mathbf{F}=\overrightarrow{\nabla V}$ is continuous on the curve C. Then:
>
> $$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \overrightarrow{\nabla V}\cdot d\mathbf{r}=V(\mathbf{r}(b))-V(\mathbf{r}(a))=V(\text{end})-V(\text{start})$$

The function V is generally called a **potential function**, and the Fundamental Theorem of Line Integrals essentially says that the work done by a **conservative vector field** in following a given path is the **potential difference**. The fact that a gradient vector field is conservative should be clear from the statement of this theorem. In the case of a gradient vector field, the work depends only on the values of the potential function at the endpoints – not on any particular path followed from the starting point to the endpoint.

Proof of the Fundamental Theorem of Line Integrals: This is just a blending of the **Fundamental Theorem of Calculus** and the **Chain Rule**. In the case of $\mathbf{F} = \overrightarrow{\nabla V}$ where $V = V(x,y)$ in $\mathbf{R}^2$, we have:

$$\int_C \overrightarrow{\nabla V} \cdot d\mathbf{r} = \int_C \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle \cdot \langle dx, dy \rangle = \int_C \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \right) = \int_a^b \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \right) dt$$
$$= \int_a^b \frac{d}{dt}(V(\mathbf{r}(t)))\, dt = V(\mathbf{r}(b)) - V(\mathbf{r}(a))$$

There are two other important aspects to this topic, namely:

(a) How do you know when a given vector field is a gradient (conservative) vector field?

(b) If you know that a vector field is conservative (gradient), how do you find a potential function?

We can partially answer the first question by invoking **Clairaut's Theorem** (equality of mixed partials). Specifically, if $\mathbf{F} = \overrightarrow{\nabla V} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle = \langle P(x,y, Q(x,y) \rangle$ where $V(x,y)$ is sufficiently differentiable, then:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial Q}{\partial x}$$

So it would have to be the case that $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$. If not, then the vector field could not be a gradient vector field.

In the case where $\mathbf{F} = \overrightarrow{\nabla V} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle P, Q, R \rangle$, there are three such relations that would have to hold:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial z \partial x} = \frac{\partial^2 V}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z}\right) = \frac{\partial R}{\partial x} \quad \Rightarrow \quad \boxed{\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial^2 V}{\partial z \partial y} = \frac{\partial^2 V}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right) = \frac{\partial R}{\partial y} \quad \Rightarrow \quad \boxed{\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

If any of these three conditions fails to be the case, then the vector field could not be a gradient vector field.

These calculations provide necessary conditions for a vector field to be conservative, but they do not provide sufficient conditions. For that we'll need either **Green's Theorem** (in $\mathbf{R}^2$) or **Stokes' Theorem** (in $\mathbf{R}^3$). However, if we can find an everywhere differentiable **potential function**, then this will be sufficient.

This brings us to the second question – finding a potential function after we have established that the conditions above have been met. This is really just a matter of finding antiderivatives and a little detective work, though often it comes down simply to "guess and check."

For example, suppose we are given the vector field $\mathbf{F} = \left\langle 2xy^3, 3x^2y^2 + 8y \right\rangle = 2xy^3\,\mathbf{i} + (3x^2y^2 + 8y)\,\mathbf{j}$. Before doing anything else, we check to see if the required condition is met:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy^3) = 6xy^2 \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3x^2y^2 + 8y) = 6xy^2 \text{, so } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ and we're good to go!}$$

We're looking for a function $V(x, y)$ such that $\frac{\partial V}{\partial x} = P(x, y) = 2xy^3$ and $\frac{\partial V}{\partial y} = Q(x, y) = 3x^2y^2 + 8y$.

The first condition implies that $V(x, y) = x^2y^3 + g(y)$ where $g(y)$ is an arbitrary function of y alone. Differentiation then gives that $\frac{\partial V}{\partial y} = 3x^2y^2 + g'(y) = Q(x, y) = 3x^2y^2 + 8y$, so we must have $g'(y) = 8y$. Therefore $g(y)$ must be of the form $g(y) = 4y^2 + C$ where C is an arbitrary constant. The potential function must then necessarily be of the form $V(x, y) = x^2y^3 + 4y^2 + C$. We actually only need one potential function, so we just take the arbitrary constant to be $C = 0$ and we use $V(x, y) = x^2y^3 + 4y^2$.

It's important to note that you could have looked at both of the components of **F** and **guessed** a potential function, but if you do this you must take the partial derivatives to **check** that it gives the correct gradient.

Calculating the work using potential functions

Example: Find the work done by the vector field $\mathbf{F} = 2xy^3\,\mathbf{i} + (3x^2y^2 + 8y)\,\mathbf{j}$ along some wild and crazy path from the starting point $(1,1)$ to the endpoint $(2,3)$. [Had we given a specific path, the method would still be the same.]

Solution: First, you have to check whether or not the vector field is conservative (gradient). If it isn't, then you have no choice but to parameterize the given path. However, in this case, we've already shown that this vector field is the gradient of the potential function $V(x, y) = x^2y^3 + 4y^2$. Therefore, by the Fundamental Theorem of Line Integrals, we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \overrightarrow{\nabla V} \cdot d\mathbf{r} = V(\text{end}) - V(\text{start}) = V(2,3) - V(1,1) = [108 + 36] - [1 + 4] = 144 - 5 = 139$$

Examples involving vector fields and paths in $\mathbf{R}^3$ work pretty much the same way except that you have to check three conditions in order to verify whether a given vector field could possibly be a gradient vector field, and then you have to use a bit more deduction or careful guessing and checking to find the potential function.