Cambridge Math Circle: What, Why and How?

Prepared for the Cambridge Public School District and the Cambridge STEAM Initiative

May 19th, 2023

Mira Bernstein, Ph.D.,
Christine Mastal, and
Nataliya Yufa, Ph.D., M.Ed.

Cambridge Math Circle

info@cambridgemathcircle.org

Introduction

There are several organizations working in Cambridge and beyond to help students feel strong as mathematicians and to support the work happening in school classrooms. The goal of this document is to provide insight into Cambridge Math Circle's work and how it fits into the Cambridge math education landscape. While the best assessment would include weeks of observing classes and interviewing stakeholders to get the full picture of our work, this document will provide a foundation for understanding our goals, methods, and results.

This report consists of two parts: Part I addresses the key ideas behind our work, and Part II describes fundamental details. We discuss all of the different types of offerings we have, because we have found that our work is most impactful when it takes into account meeting students where they are, and every class presents a unique combination of skills, aptitude, and ability.

Table of Contents

Part I:
Introduction to Cambridge Math Circle's Work

Background - History of Math Circles

Math circles originated in Eastern Europe in the twentieth century. The essence of a math circle is to bring together like-minded people who enjoy math, to discuss and solve complex math and logic problems together and to build community around mathematics. There are dozens of books written on mathematical topics that are usually taught at math circles, while the exact curriculum depends on the interests and background of the students and teachers. In general, topics covered are complementary to the school curriculum, so that both kids who are "strong"[1] in school math and those who struggle can have access: math circles are the ultimate low-floor, high-ceiling collections. Areas of math include many that are normally taught only at the college level: combinatorics, topology, graph theory, and logic, among others.

Currently, there are over 100 math circles in the United States, which are part of a math circle network, organized by the American Institute for Mathematics (AIM). Cambridge Math Circle partners with several of the largest math circles throughout the country, including the Math Circles of Chicago, the Prime Factor Math Circle in Seattle, and the Ithaca Math Circle (out of Cornell University).

Background of Cambridge Math Circle (CMC) Co-Founders

Nataliya Yufa holds a bachelor's degree from MIT in mathematics and physics, and a Ph.D. in physics from the University of Chicago, as well as an M.Ed. from Lesley University in Math Education. Her career as an educator began while she was a Ph.D. student in Chicago, leading teams of other graduate students and postdocs teaching science in afterschool clubs in Chicago Public Schools. Her experiences inspired Nataliya to become a full-time math educator. She has taught at several Boston-area schools, including Cambridge Public Schools, and has observed dozens of classrooms across many schools, learning what works.

Mira Bernstein received her PhD in math from Harvard University, taught at Wellesley College for six years, and then left academia to pursue her dual interests in data science and math enrichment education. She co-runs Canada/USA Mathcamp, an international summer program for mathematically talented high-school students, and helped found Proof School, a school for kids who love math in San Francisco. Mira's work at a summer math program for underserved middle-school students was featured in the New York Times.

[1] In the interest of brevity, we define students who have a positive math mindset and are able to persevere and solve complex, new-to-them problems, as "strong." Alternative language is "students ready for more challenge in math" or "students who have experienced success in math class." We use these terms as synonyms.

CMC Within the Overall Cambridge Math Space

The chart below details the overall extra-curricular math space in Cambridge. Most programming, whether it targets struggling or strong students, focuses on the Middle-Class+. Middle-Class+ programs require both a higher income and the ability to transport students to nearby towns for in-person classes.

Cambridge Math Circle is the only program that specifically targets struggling AND strong students, with a focus on low-income students, girls, and Black and Latinx students.

Chart of Extra-Curricular Math Programs in Cambridge

Math Program	Grade Levels	Focus (instruction, inspiration, remediation)	Income Target	Math Achievement Target Group	Cambridge Based?
MathTalk	JK - 3	Early math	All; focus on low-income	n/a	yes
Cambridge Math Circle	**1-8**	**Remediation via enrichment, and pure enrichment**	**All; focus on low-income**	**All students, with a focus on students in the top 75%**	**yes**
Young People's Project	4-8	Remediation; inspiration for math teaching careers, building community	All; focus on low-income	Students in the bottom 25%	yes
Tutoring Plus	4-12	Remediation/academic support	All; focus on low-income	Struggling students	yes
Girls' Angle	5-12, girls only	Inspiration and enrichment, building community	Middle class+	"Strong" students	yes
Studio of Engaging Math	JK - 8	Remediation and some enrichment	Middle class+	Struggling students and on-grade students	no (Boston)
Mathemagics	1-12	Remediation enrichment	Middle class+	Struggling and "strong" students	no (online only)
Kumon	K-12	Remediation academic support	Middle class+	Struggling students	no
Mathnasium	K-12	Remediation/academic support	Middle class+	Struggling students	no
Russian Math School (RSM)	K-12	Preview, some rote learning, some remediation, some inspiration	Middle class+	Struggling, on-grade and "strong" students	no
AoPS Academy	2-12	Preparation for STEM majors and careers	Middle class+	"Strong" students	no

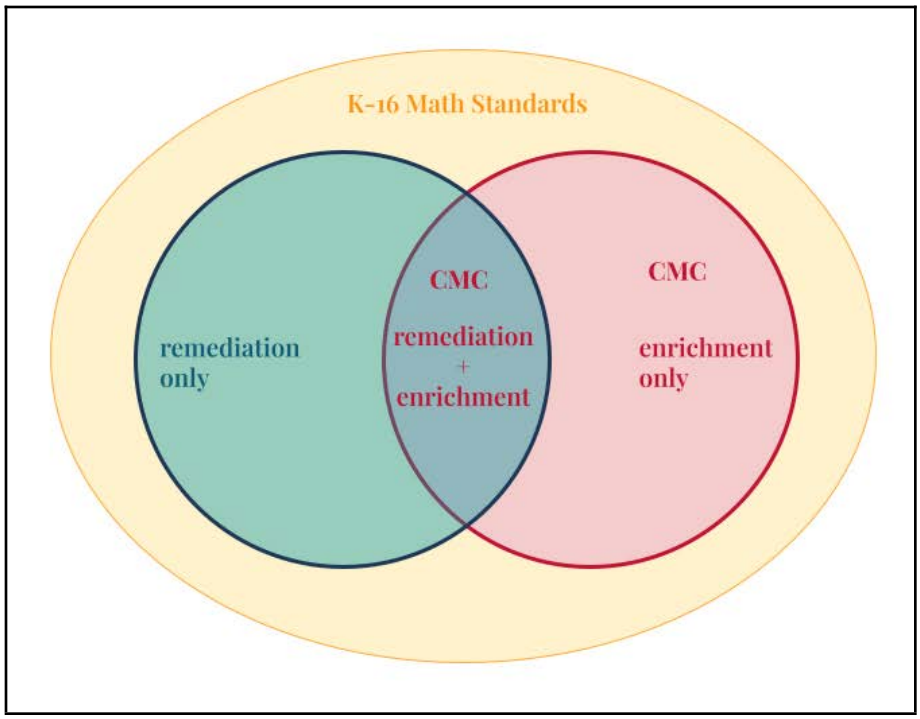
What Is the Cambridge Math Circle and Who Is It For?

The mission of the Cambridge Math Circle (CMC) is to create a welcoming space for elementary and middle school students to explore fun, challenging math. CMC works with all students, with a special focus on those from groups underrepresented in STEM fields.

Based on the mission, **CMC is for any student who wants a deeper connection to math, needs a reset with their relationship with math, and/or wants more rigorous work to prepare them for honors and AP classes in high-school.** As an equity-focused organization, we reach 75% or more underrepresented students in STEM, including: girls and non-binary students, African-American, Latinx or Indigenous students, and/or low-SES students.

Our program complements the work already happening within the general math classroom and increases access to additional quantitative, logical, and problem-solving skills for all students. Programs like CMC are often an experience only available to kids with parents already in STEM, or high-income families that see the value in STEM and have the means to pay for private programs. Our goal is to change this status quo.

Is CMC a Remedial or an Enrichment Program?



As the diagram shows, CMC engages both in pure enrichment and in remediation combined with enrichment. **Our work resides in K-16, ie both K-12 and college content standards.** Most programs we are aware of engage in only pure remediation or only pure enrichment. We

work in the space where the two overlap, in addition to pure enrichment, because of the needs of our students.

Many students in our target groups need pure focused remediation to get them caught up and able to master the key grade-level standards. However, when students receive remediation only, it negatively affects their math mindsets and ability to see themselves as successful in mathematics. There are many excellent remediation-only interventions. Therefore, **CMC provides either pure enrichment or a remediation + enrichment combination, so that students can believe they can be successful in math *and* also achieve success in math.**

By bringing together kids who are already "strong" in math and those who aren't "strong" yet, we create a space where being "bad at math" is not a requirement. This helps elevate the classroom status of students who struggle in school math class, by showing them (both to themselves and their peers) as working at and being successful in, on their own time, challenging math tasks.

Goals and Teaching Approach of the CMC Program

The goals of CMC are **to help raise mathematically-competent citizens** and **to help more underrepresented students pursue STEM careers**.

Based on these long-term goals and on our experience in CPS and other schools (first-hand and from informal interviews with teachers, coaches, and parents, as well as working directly with students out of school time), we have identified **five fundamental areas to help Cambridge students succeed, outlined on the next page,** which are incorporated into both types of classes.

Further, CMC uses two types of curricula for its classes:

A) a custom-created Math Circle curriculum and
B) the Beast Academy/Art of Problem Solving curriculum

The five areas of focus are the same for both curricula, but the emphasis is somewhat different. (See Part II for more details on each of the curricula.)

Five Fundamental Areas to Support Cambridge Students' Success:

Focus Areas	Goals	A. Priority in Math Circles	B. Priority in Beast Academy/ AoPS Classes
I) Problem Solving: Build up problem-solving skills for unusual and/or multi-step problems	For success in middle school, high-school, college and careers.	high	high
II) Grit: Increase students' grit and resilience in the face of unfamiliar problems	For success in every future math class and for everyday life.	high	medium
III) Mental Math: Improve computational fluency by explicitly teaching mental math techniques based on conceptual understanding	To remove roadblocks to success in middle school and high school math classes.	medium	high
IV) Deep Previewing: Introduce algebraic and geometric fundamentals, as well as ideas of proofs	To ensure success in honors and AP level high school classes.	medium	high
V) Joyful Learning: Build up the enjoyment of math	Both to ensure interest in mathematics and STEM careers, and to enable more effective learning.	high	medium

I) Problem-Solving Skills and **II) Grit and Resilience:**

Why focus on Problem-Solving and Grit?

Many students assume that if they can't solve a problem right away, then there's something wrong – they must not have understood the material. But that's not how advanced math works. Creative problem-solving requires a separate set of skills: approaching a problem from multiple directions until it yields, looking at small examples and special cases, etc. Success in problem-solving makes students less afraid of not knowing the answer, more willing to experiment, and builds resilience.[2]

How does CMC teach problem-solving skills, grit and resilience?

[2] Boaler, Jo. *Mathematical mindsets: Unleashing students' potential through creative mathematics, inspiring messages and innovative teaching*. John Wiley & Sons, 2022.

By giving *all* students hard problems that only require the technical skills they learned in elementary school math. We review the topic together, and guide students through the process of solving deep, challenging problems. This sparks their interest in math because they get the satisfaction of solving a hard problem on their own, rather than constantly struggling to understand a new topic.

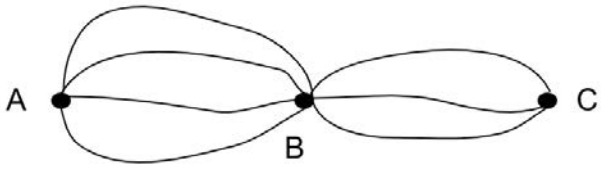
For example, in the problem set below (from our Math Circle curriculum), the problems look superficially different, but they can be solved using the same method. At first, most students do not know how to approach the problem: we encourage them to try tracing different paths or make a list of the different kinds of pizza. By working with examples, they come to understand the pattern and the fundamental conceptual similarity of the two problems.

Problem 1.

a) There are 4 different types of toppings and 3 kinds of pizza bases at Armando's pizzeria. How many different pizzas can we order if each pizza only has one topping and a base?

b) Now Armando's added two different kinds of sauces. If we have a pizza consisting of a topping, a sauce and a base, how many different pizzas can we order now?

Problem 2. a) There are 4 roads leading from Acton to Bedford, and 3 roads from Bedford to Cambridge, as shown in the picture. If Elias wants to bike from Acton to Cambridge a different way each time, how many different ways are there?



How does teaching problem-solving, grit and resilience align with school math?

Nowadays, virtually every teacher is aware of the importance of grit and resilience for learning outcomes. Helping students develop grit is not easy, especially while covering the grade-level curriculum and only seeing each student for a year. Because of the setup of our program, we frequently get to work with students for multiple years. We can continue to push them to grow and become more resilient thinkers because we can reference newly acquired skills over time, and point out how much they've grown.

Because schools have a large number of standards to cover each year, many students, especially those who struggle, may not have time to engage deeply with every topic during the regular school math class. Our classes *can* take more time with challenging problems, since that is the only focus for the lesson. Students have reported in multiple surveys that their experiences in our classes increase their confidence in *school* math classes.[3]

III) Mental Math

Why focus on mental math?

Arithmetic is fundamental to virtually all of mathematics. In particular, a strong background in arithmetic makes it a lot easier for students to learn algebra, since a lot of algebra is arithmetic with variables. By revisiting grade standards from *prior* years *and* going into greater depth on conceptual understanding of arithmetic than school class-time allows, we are helping students become comfortable with many different ways of looking at the same computation.[4] This is crucial for algebra: not just being able to solve an arithmetic problem, but to be able to play with numbers in your head.

How does CMC do it?

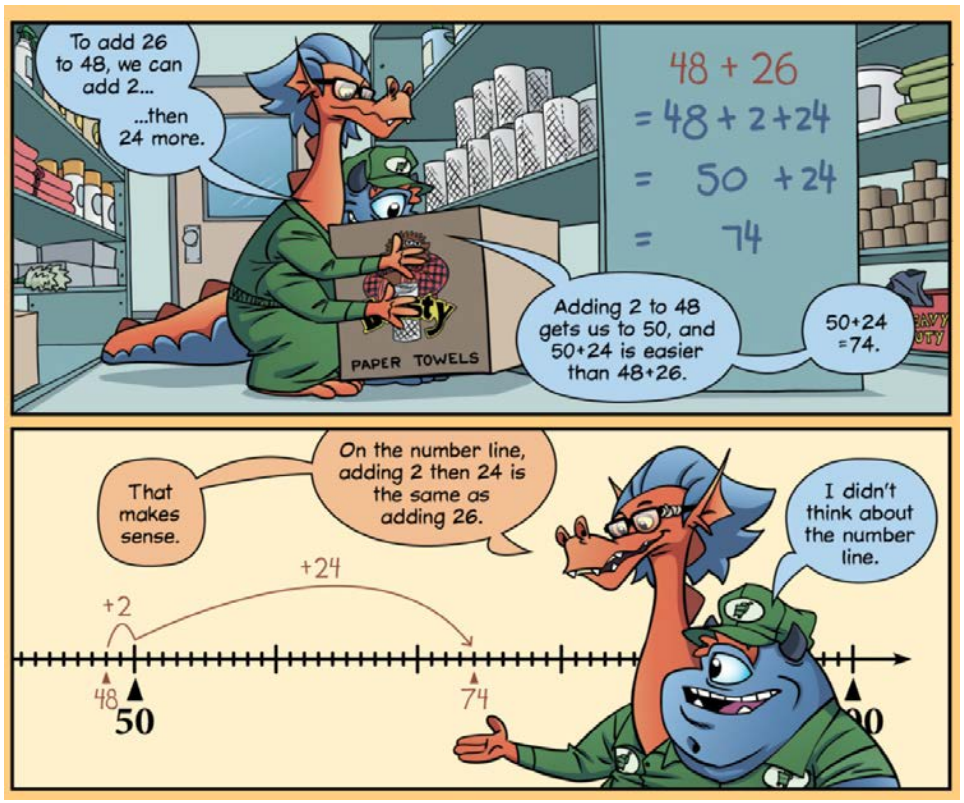
We start with an intuitive explanation of why a particular technique works, then give students time to build up fluency. However, to prevent fluency from turning into rote memorization, we continue to insert problems where the basic technique does not work, so that students have to keep revisiting the basic ideas behind the intuition. In this approach, deepening conceptual understanding and building computational fluency go hand in hand.

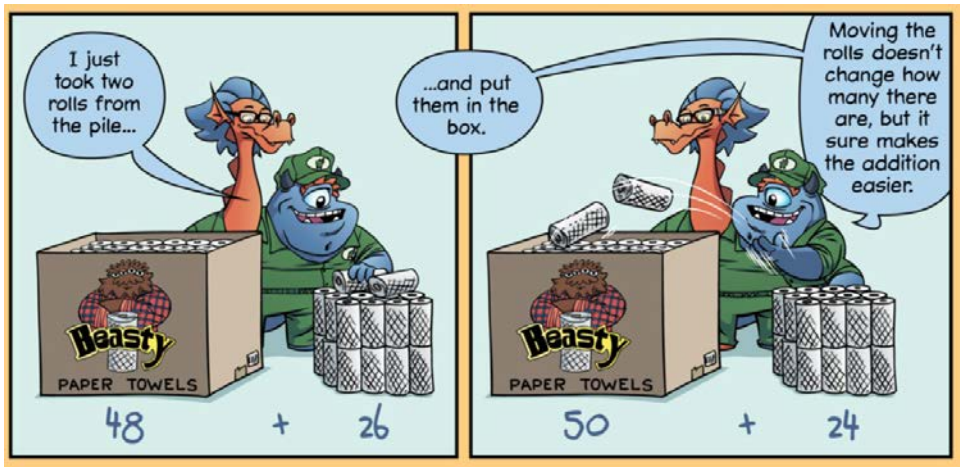
Example 1. Addition strategies (Beast Academy, 3rd grade, taught to 2nd-7th graders at CMC, as needed)[5]

[3] 80% of surveyed CMC students in 2021 reported increased confidence in math class as a result of either math circle or Beast Academy classes.

[4] Every teacher has seen this happen: you teach, kids do great, and then a few weeks later it's like the lesson never happened. Revisiting the standards from prior years, and teaching them with colorful characters, allows students to really internalize all of the tools they may have seen and forgotten, or missed because they weren't in school that day, etc.

[5] Beast Academy curriculum, BA2, Addition.





Beast Academy problems start out simple and build in difficulty.

The first problem in the first section is the most straightforward.
(The following problems are all from the same chapter, BA2, Addition.)

Fill in the blank to make a true statement.

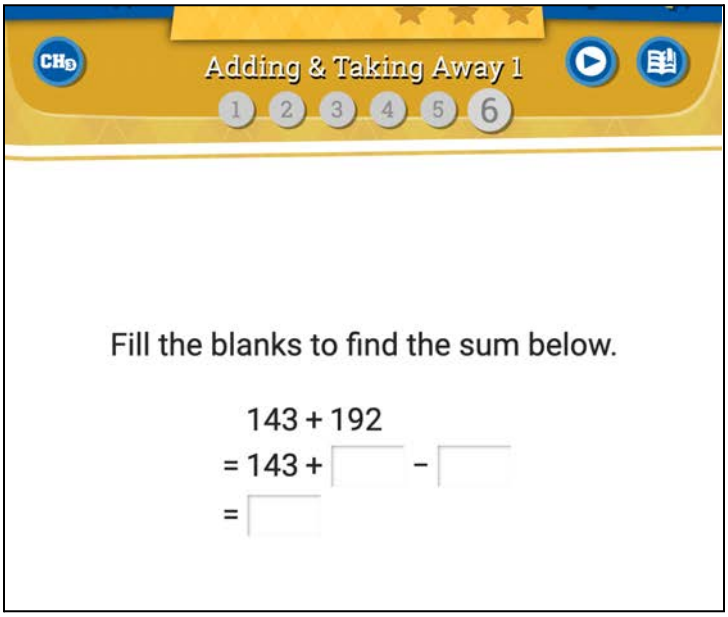
Adding 78 to a number is the same as adding 80, then taking away ☐ .

The middle problem in the first section is a little different.

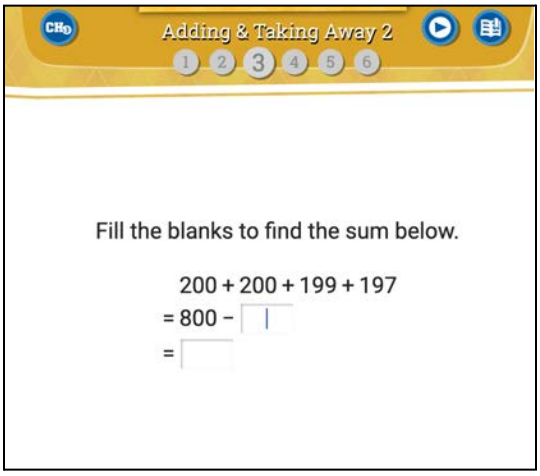
Fill in the blank to make a true statement.

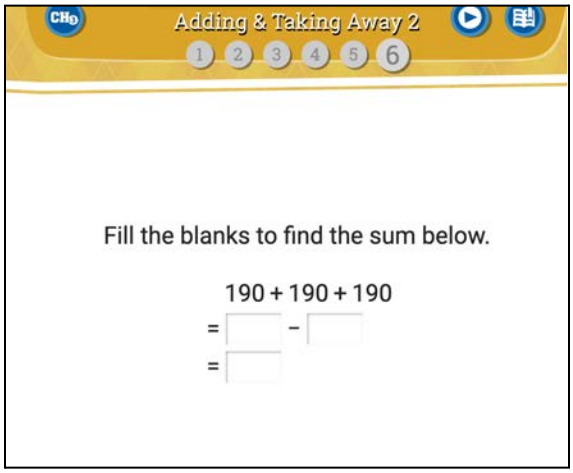
Adding ☐ to a number is the same as adding 100, then taking away 7.

Subsequent problems extend the same idea to larger numbers and to adding more than two numbers. The general idea is that students cannot turn their brains off and do the same thing over and over, but need to think.



In recognition of the many ways that kids' attention is being grabbed in today's world, most of the problem pages are very "clean," without pop-ups or animation to distract students.





How does this work align with what's being taught in school?

We have not encountered any teachers of mathematics who wouldn't want their students to have stronger mental math skills. While school math class allows for some of that work, for many students, the need to focus on the new conceptual material being taught takes away from their ability to strengthen mental math, too. We can give kids challenging problems on topics that are review for them (e.g. addition for 3rd graders, or fractions for 5th and 6th graders) to help them achieve both conceptual understanding and automaticity when it comes to mental math, so that they are not held back from success in their grade-level math by computational mistakes.

IV) Deep Previewing: Introducing algebraic and geometric fundamentals, and proofs

Why focus on Deep Previewing?

Previewing distant topics (those coming much later) is good both for students who are ready for more of a challenge by meeting them where they are, and for students who are behind grade level. We've found that for students who are behind, even though they're missing some material, they can be cognitively ready to understand the abstract ideas behind these topics, which helps keep them interested in the subject. Research shows that previewing is beneficial for struggling students' self-beliefs especially, as it allows students to succeed the first time, rather than succeed after a failure.[6]

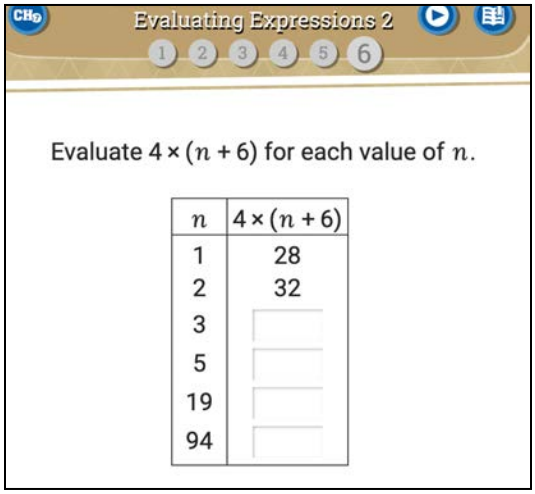
[6] , J.P. and Miller, R.H. (2006) 'Effects of pre-teaching and re-teaching on math achievement and academic self-concept of students with low achievement in math', *Education*, 126(4), 747+, available: https://link.gale.com/apps/doc/A149157232/AONE?u=mlin_oweb&sid=googleScholar&xid=6d5c5567 [accessed 26 Apr 2023].

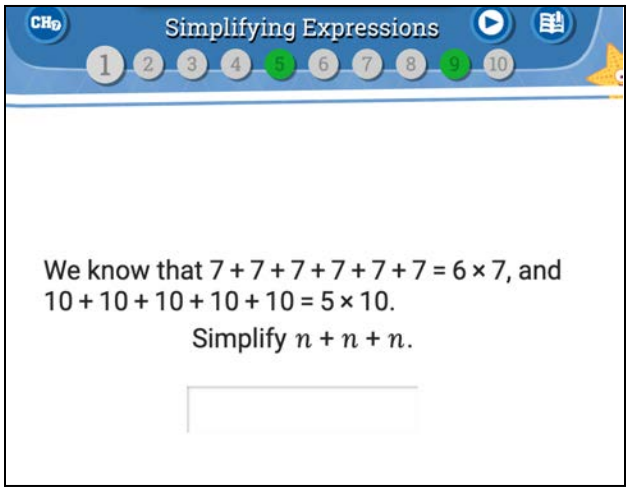
Every elementary and middle school student we have met gets excited when we tell them that they are learning topics that are usually taught in high school or college, especially if the students are normally considered to be behind. So both a 3rd grader and a 6th grader who is previewing algebra in BA4 will benefit, only in different ways. For the 3rd grader, this previewing is giving them the additional challenge that they crave; for the 6th grader, it's building up "mathematical fitness," which leads into items III) and IV) problem solving strength and grit and resilience.

How does CMC preview algebraic and geometric fundamentals, and ideas of proofs?

a) **Algebraic Fundamentals:** In school math classes, students learn about variables and solving equations. We also know that students often get tripped up with algebraic manipulations in high school through our own teaching experiences, and through our colleagues teaching honors or AP math courses. To address this, we spend significant time on manipulating algebraic expressions, and deeply understanding the distributive property. This approach goes beyond just the idea of variables and using letters for unknowns and gives students a more intuitive understanding of how algebra works.
(Material below is from Beast Academy's BA4, Introduction to Variables chapter.)

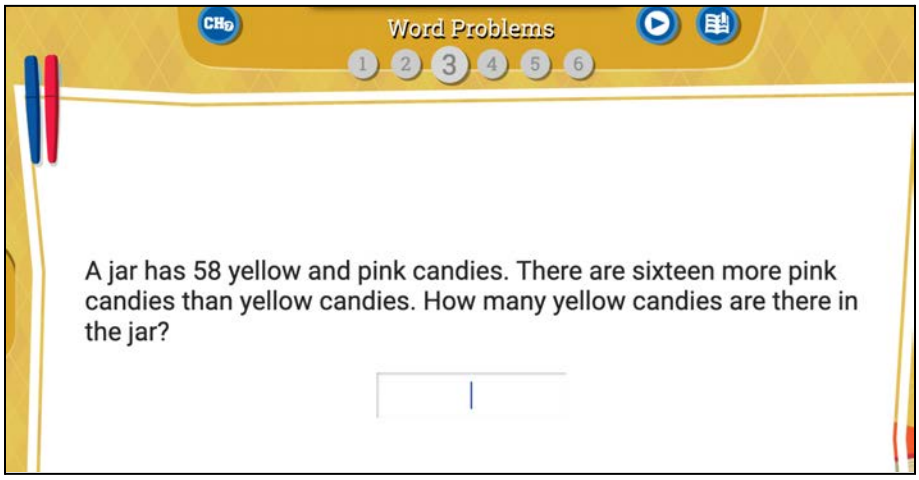
Examples of problems using variables:





Example of an excerpt from the comic storyline on variables:

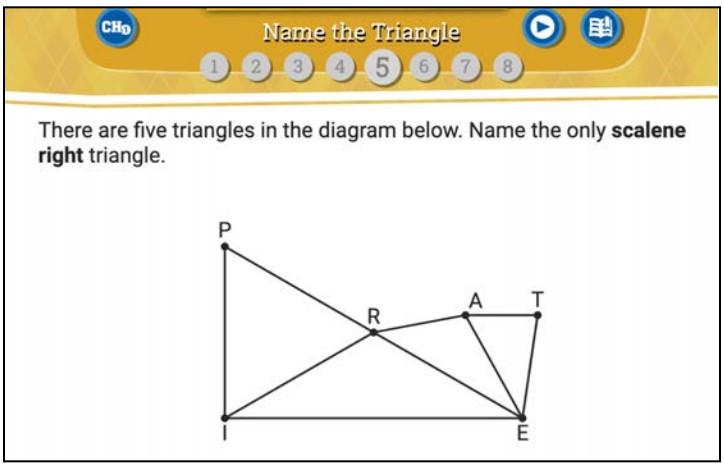
Example of a word problem that includes variables:

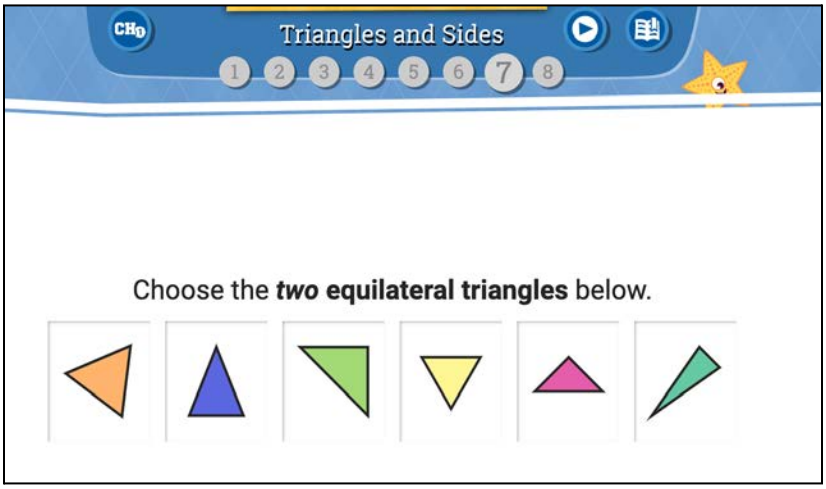


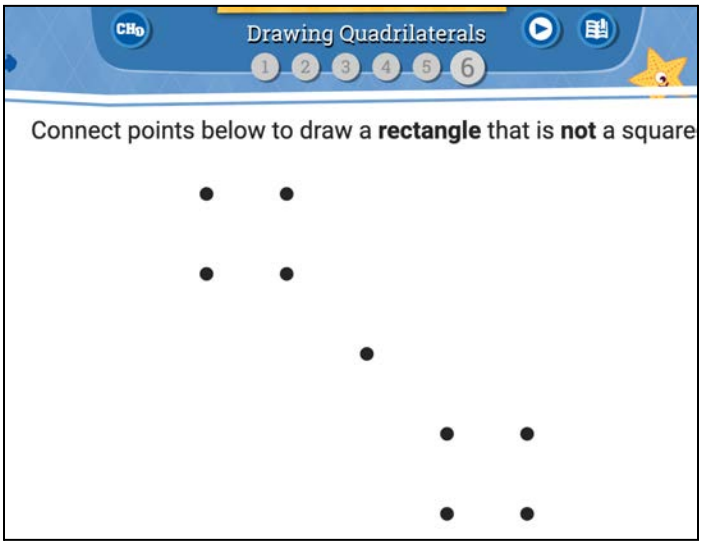
b) **Geometric Fundamentals:** Many students find geometry bewildering. Especially when it comes to proofs, we have seen first-hand the struggles students have when they don't know whether all rhombuses are squares or whether all squares are rhombuses. Definitions of lines, rays, segments, and other geometry concepts can be overwhelming to students when they are presented all at once.

By having students solve hard problems involving these basic definitions, we are preparing them to be solid in their prerequisites for high school geometry. By doing this in the years when these topics aren't part of the grade standards (between initial instruction in elementary school math class and when the definitions are used in high school geometry), we are making sure students have both their definitions and strong background in logic, needed for the rigor of geometry, calculus and other courses.

Examples of geometry problems:



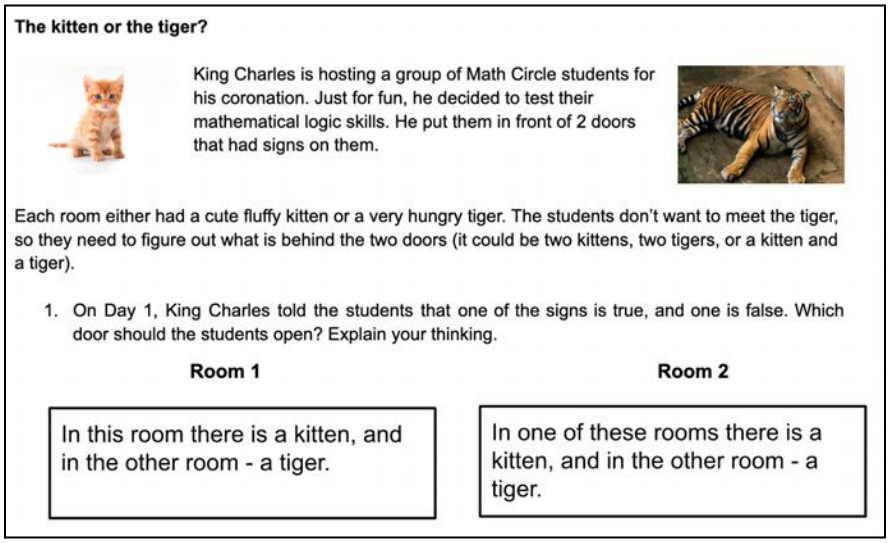




c) Proofs:

Proofs are based on logic; by teaching different logic concepts early-on, we are preparing students for future work with proofs. For example, when students solve kitten or tiger problems, or knights and liars problems in math circles, they are building up their logical thinking skills, just as they are in solving many of the puzzles and problems in Beast Academy/Art of Problem Solving. Many of our math circle problems require proofs as solutions. To keep things low-key, whenever we require a proof, it is called "Explain your thinking." We work with kids at the level of proofs that they are capable of and push them a little more each time.

Examples of work with proofs from the Math Circle curriculum.



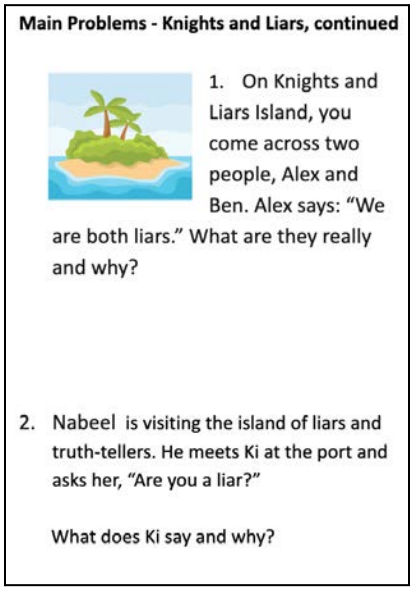
The kitten or the tiger?

King Charles is hosting a group of Math Circle students for his coronation. Just for fun, he decided to test their mathematical logic skills. He put them in front of 2 doors that had signs on them.

Each room either had a cute fluffy kitten or a very hungry tiger. The students don't want to meet the tiger, so they need to figure out what is behind the two doors (it could be two kittens, two tigers, or a kitten and a tiger).

1. On Day 1, King Charles told the students that one of the signs is true, and one is false. Which door should the students open? Explain your thinking.

Room 1	Room 2
In this room there is a kitten, and in the other room - a tiger.	In one of these rooms there is a kitten, and in the other room - a tiger.



Main Problems - Knights and Liars, continued

1. On Knights and Liars Island, you come across two people, Alex and Ben. Alex says: "We are both liars." What are they really and why?

2. Nabeel is visiting the island of liars and truth-tellers. He meets Ki at the port and asks her, "Are you a liar?"

   What does Ki say and why?

How does Deep Previewing align with what's being taught in school?

Most school curricula preview algebraic and geometric fundamentals, often introducing variables as early as 2nd grade. Our work in Deep Previewing complements school math by taking deeper dives into the topics for which there usually isn't time during math class, building a smoother transition between elementary and middle school, and preparing for the rigor of high school honors and AP math classes.

V) Joyful Learning

Why focus on joyful learning?

All of us learn more and learn better when we are enjoying the learning.[7][8] At CMC, we think it's important to bring fun and joy to math to help spark a child's joy in the material.

[7] Willis, Judy. The Neuroscience of Joyful Education, *Engaging the Whole Child,* Summer 2007, Vol. 64.
[8] Hirsh-Pasek, Kathy, Roberta Michnick Golinkoff, Kimberly Nesbitt, Carol Lautenbach, Elias Blinkoff, and Ginger Fifer. *Making schools work: Bringing the science of learning to joyful classroom practice*. Teachers College Press, 2022.

Every math teacher wants their students to experience joyful learning, and of course, it happens during the school day, which is great. We know from our own school teaching experience, however, that when there's a curriculum to cover, there are going to be topics that many children do not enjoy, such as memorizing addition facts, practicing long division or graphing lines.[9][10] Many students decide they cannot possibly learn math because they struggle with these and other topics.[11] By giving students additional joyful experiences in math, while enlarging their view of what constitutes math, we are helping them to bring a positive attitude into their math classrooms, as detailed in the section on "Alignment to work in math classrooms" below.

How does CMC facilitate joyful learning?

In our experience, the slightly more relaxed atmosphere outside of class time allows for greater communication and increased creative freedom. Students find joy in math when they can talk to their partners or groups, and when tasks are varied and enjoyable. Most importantly, this includes many students who previously did not think they liked math.

One of the ways we do this is by expanding students' ideas of math. In our classes, students learn that geometry and topology, Sudoku and KenKen, logic and combinatorics are all types of math. If they struggle with arithmetic but discover they are great at building symmetrical 3D shapes, they realize that they aren't "bad at math," they just need a little more work in one area.

How does CMC's joyful learning align with work in math classrooms?

By taking the time to allow students to explore topics that are intrinsically motivating, we are helping students find joy in math, which they then take to the rest of their lives, including math class at school. **In fact, 84% of the students surveyed said that they enjoyed their *school* math class more as a result of their CMC experience.**

[9] Willis, Judy. *Learning to love math: Teaching strategies that change student attitudes and get results*. ASCD, 2010.
[10] LaMar, Tanya, Miriam Leshin, and Jo Boaler. "The derailing impact of content standards–an equity focused district held back by narrow mathematics." *International Journal of Educational Research Open* 1 (2020): 100015.
[11] NY once watched a 6th-grader attempt a problem during school math class that required multiple steps, with operations such as addition and multiplication followed by long division. After making several mistakes and ending up wildly off with his answer, the student proclaimed "I'm bad at math!" The way many math classrooms are set up, students who aren't capable of accurate arithmetic by middle school often do not get a chance to get better, because there is so much grade-level content to be covered. They fail again and again, leading to the conclusion "I'm bad at math!" NY has had several summers of working with middle schoolers and high schoolers, in Cambridge and Brookline, convinced that they were bad at math because they couldn't accurately compute.

Part II: Details of Cambridge Math Circle's Work

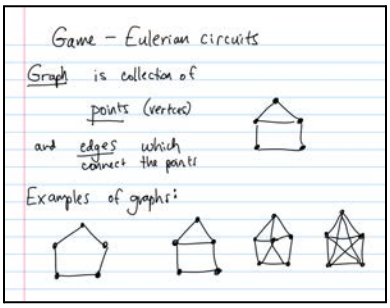
CMC Curricula

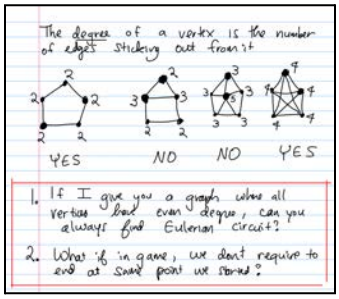
Math Circle Curriculum

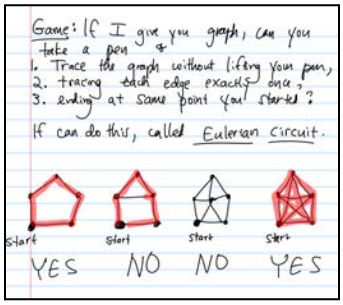
Our Math Circle curriculum is similar to other math circles around the U.S. and in other countries. It includes topics complementary to what is taught in school at every grade level, creating a more level playing field. This gives a chance to shine to kids that aren't necessarily strong in traditional school math, while giving kids who are strong in traditional school math something that they can productively struggle with.

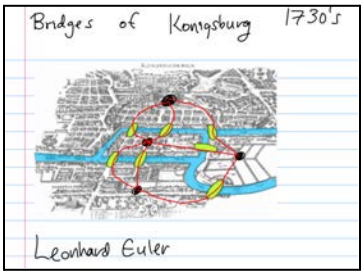
Topics include combinatorics, logic, graph theory, topology, algebraic thinking. We also include problem-solving techniques, such as drawing a picture, working backwards, proof by contradiction, and case analysis, among others. The problems are modified and selected to fit the needs of the students currently in the class.

We draw on our collection (examples below) of approximately fifty different math circle and math enrichment books, as well as curricula shared with us by The Math Circles of Chicago and The Prime Factor Math Circle of Seattle. We also host guest instructors from many different organizations: Harvard, MIT, Williams College, and more, and subsequently incorporate their lessons into the curriculum.[12]









[12] This sample material on graph theory was prepared and originally taught by Prof. Lauren Williams of Harvard University.

Beast Academy/Art of Problem Solving Curriculum

For our Beast Academy (BA) and Art of Problem Solving (AoPS) classes, we use the BA/AoPS online curriculum with support from a CMC teacher. We facilitate a collaborative environment within each small group, allowing students to share ideas and solve the problems as a community, which is how we carry the spirit of a math circle through this curriculum.

Grades 3-6

Materials for grades 3-6 focus on simultaneously building up computational fluency, conceptual understanding, problem-solving techniques, and perseverance in mathematics, while also going in depth and previewing content from middle school and high school. Each student is individually evaluated to see what strengths and weaknesses they have.

The youngest students usually start with pirate numbers - a version of the Roman numeral system that is actually the decimal system in disguise. This helps students build a deeper understanding of our decimal system, while not repeating things that they've already done in class, such as ten-blocks.

Older students in the 3-5 grade range will often start with geometry, for two reasons:

a) it can be an area of weakness for students strong in arithmetic and algebraic thinking,
b) it can be an area of relative strength for students who have a history of struggling in math class. [13]

Starting with geometry allows kids who haven't experienced success to experience it, and those who've always felt strong to have something to struggle with.

After geometry at the appropriate level, students move on to either a multiplication deep-dive, or division, fractions, decimals, and exponents, as appropriate.

Beast Academy is ideal for supplemental math because every problem is a little different (and usually a little harder) than the one before. Rather than following the same steps over and over, students need to think before applying the techniques, resulting in stronger understanding and retention. The content is presented via comics, which the kids find

[13] Students can be broken into 4 rough groups at any point in time (these aren't static assignments, and our job as teachers is to help them to move from some of these groups into others): AG, AnG, nAG, nAnG. Here A and G stand for having the skills and mindset to succeed in Algebraic concepts and Geometric ones, respectively, and "n" in the front means not being ready for a particular concept immediately following. "nAG" means not ready for Algebraic concepts, but ready for Geometric ones. For students who are equally ready (or equally not ready) for Algebraic and Geometric ideas, it makes no difference with which one to start the year. For students in the divergent categories (nAG or AnG), we have a chance to reverse the script. Most of elementary math is concerned with computation and Algebraic concepts. By switching to geometric ones at the start of the year, we are giving a significant subset of kids the opposite of what they are used to - a challenge for those who aren't used to it, and success for those who aren't used to it.

entertaining and enjoy reading out loud. The comics show characters struggling, failing, and eventually succeeding, providing authentic inspiration.

Grades 6-8

Material for grades 6-8 builds on the foundation from the earlier grades. Students are again assessed to find their strengths and weaknesses, and are assigned to groups accordingly. We often have multiple groups at the same level, based on students' ages and personalities; for instance, shy students are generally grouped with other shy students, so that they can learn to speak up and then move on to more mixed groups. Curriculum covered varies from fractions, ratios, and percentages, to Art of Problem Solving Prealgebra and Algebra.

Below are a few sample problems with solutions from the Art of Problem Solving Algebra course.[14]

7.2.2:

Five people can mow a lawn in 12 hours. How many more people are needed to mow the lawn in just 3 hours, assuming each person mows at the same rate?

Solution: The number of people mowing and the time required to mow are inversely proportional. Letting n be the number of people and t be the amount of time, we have $nt = (5)(12) = 60$ because 5 people can mow a lawn in 12 hours. If m people can mow the lawn in 3 hours, then we must have $m(3) = 60$, so $m = 20$. Therefore, we need to add $20 - 5 = \boxed{15}$ people to the job.

7.3.3:

The force of the gravitational attraction between two bodies is directly proportional to the mass of each body and inversely proportional to the square of distance between them. If the distance between two bodies is tripled and the mass of each is doubled, what happens to the force of gravitational attraction between them?

Solution: Because the force of gravitational attraction is directly proportional to the mass of each body, when the mass of one body is doubled, the force of gravitational attraction between them is doubled. Therefore, when the masses of both bodies are doubled, the force of gravitational attraction between them is doubled twice, so it is multiplied by 4.

Because the force of gravitational attraction is inversely proportional to the square of the distance between the two bodies, the product of the force and the square of the distance between the two bodies is constant. So, when the distance between the two bodies is tripled, the square of the distance is multiplied by 9. Since the product of the force and the square of the distance between the bodies is constant, when the square of the distance is multiplied by 9, the force must be divided by 9.

Therefore, if the distance between two bodies is tripled and the mass of each of the bodies is doubled, the force is divided by 9 to account for the change in distance and multiplied by 4 to account for the change in mass. So, the force is $\boxed{\text{multiplied by } 4/9}$.

8.3.4:

Let point A be $(2, 7)$ and point B be $(-6, -3)$. What point is on the segment connecting A and B such that the distance from the point to B is 4 times the distance from the point to A?

Solution: Call the new point C. As we did with the similar problem in the text, we handle the two coordinates separately. To get from A to B, we go 8 steps left and 10 steps down. Suppose we must take x steps left to get from A to C, so we must take $8 - x$ steps left to continue from C on to B. Since C is 4 times as far from B as it is from A, we must have $4x = 8 - x$. Solving this equation gives $x = 8/5$. So, C is $8/5$ steps left of A. Therefore, its x-coordinate is $2 - 8/5 = 2/5$.

Let C be y steps below A, so we must take $10 - y$ steps to continue on from C to point B. Again, since C is four times as far from B as it is from A, we have $4y = 10 - y$. Solving this equation gives $y = 2$, so C is 2 steps lower than A. Therefore, its y-coordinate is $7 - 2 = 5$.

Putting our two coordinates together, we see that C is $\boxed{(2/5, 5)}$.

Suppose A is (x_1, y_1) and B is (x_2, y_2). See if you can follow the argument above to find formulas for the coordinates of the point C on $\overline{AB}$ such that $AC/BC = k$, for any positive constant k.

[14] Art of Problem Solving, *Introduction to Algebra* textbook.

Equity and Culturally-Responsive Teaching

How does CMC manage to reach all students and teach all students, with a focus on underrepresented populations?

Equity was the reason CMC was founded and it is an integral part of our mission. From teaching at the high school level (in Braintree, Brookline, Melrose, Wakefield and San Francisco, and visiting classrooms at CRLS, BB&N, and Newton South), as well as experience in running/visiting programs for high schoolers, we have seen that students who are equally talented may arrive at high school up to 5 years apart in terms of preparedness to study challenging math. We found that it is often the case that lack of peers who share an interest in math, and lack of support from outside of school, cause students in our target populations to not pursue the math necessary for college STEM classes and consequently not pursue their dream careers.

This is why CMC approaches math enrichment with a focus on building community and letting every child succeed by setting up a level playing field. For example, when we taught an enrichment class pre-COVID to a group of students at Rindge Avenue Upper School, half the students were selected by teachers as being close to passing the MCAS for the first time, and the other half were strong students who self-selected. Students gravitated towards their friends, and it would have been easy to give them different tasks based on their current skills. Instead, we worked to find topics where both groups could be successful, and offered more of an introduction to the first group. Everyone ended up working on the same topics, and when it came time to games and puzzles, students were engaging across groups.

One of CMC's co-founders, Nataliya Yufa, was trained by Manuel Fernandez, the current Chief Equity Officer of Cambridge Public Schools, in culturally responsive teaching. Through teaching in a wide variety of settings and countless conversations with students, parents and teachers, CMC continues to work on respecting each student's cultural background. The cross-section of students who participate in our activities (reflective of the schools we are in and the city as a whole) shows us we are teaching in a culturally responsive way, since our students and families have to opt in to participate.

Methodology

How does CMC simultaneously work with kids who struggle and kids that need more challenging work? We do this differently depending on the class, as described below.

Beast Academy/AoPS Methodology

The reason our differentiation works is because we are able to teach at many different levels, grouping kids together based on their math needs, skills, and personality. For example, if one 7th grader needs to work on addition and multiplication and solving multi-step problems, he or she works with 5th and 6th graders in the same place. And if a 4th grader is many years ahead and needs Prealgebra, he or she can work with 6th and 7th graders who are working on that.

Math Circle Methodology

In our Math Circles, we work on extra-curricular topics, so it is par for the course if some students solve 4 problems out of 7 while others solve 7 out of 7. We increase difficulty so that the final problems are significantly more challenging than the initial ones. We also give students who need more time a chance to investigate the topic at their own pace, by giving bonus questions and optional homework. Often the bonuses will turn up in team competitions later in the class. When possible, we group students with those who work at a similar pace, so that no one feels rushed or like they have to move at a slower pace.

Program Snapshot: Beast/AoPS

Student Profile of Beast/AoPS Program

Students in grades 2-8, with most of the students being in grades 3-6. We have students who are in one more of the following categories:

a) **students who are looking to fill-in the gaps in their knowledge**, so they can succeed in future math courses,
b) **students who are at grade level and** know that they need to know math extremely in depth because they **want to follow a STEM career**,
c) **students whose families are in underrepresented or low-SES groups**, whose families believe that they need to be much stronger than average in order to overcome these obstacles.
d) **students who love math and are looking for more challenges**.

Goals of the Beast/AoPS Program

As described in Part I, our Beast Academy/AoPS classes focus on different aspects of math learning in order to ensure we are raising mathematically competent citizens and diversifying the STEM pipeline.

Outcomes of the Beast/AoPS Program

Students in our Beast Academy classes solved problems as a whole class and also individually. The table shows the gradual switch made as we shifted away from doing many problems as a whole class and not assigning homework, to having more individual work.

At the beginning of the pandemic, in 2019-20 and 2020-21 academic years, classes met twice a week, because that's the need that our families expressed: more time for warm human interaction around math. Starting in 2021-22, we switched to once a week plus homework and office hours model, as families pointed out that once afterschool activities, such as sports, resumed, it was hard to reserve two days specifically for math. For the 2022-23 school year, we project 524 problems per student, based on the 8 months of data we have thus far.

Academic Year	**Number of months**	**Number of students (BA+AoPS)**	**Total number of problems solved (BA)**	**Average number of problems solved per student (BA)**
2019-20	3 months	37	8,317	225
2020-21	10 months	78 (18 for a part-year pilot)	17,684	263
2021-22	9 months	67 students	29,163	511
2022-23	9 months*	100 students (20 part-year)	36,843	524*[15]

While it is great to see how many problems students solve individually, true change has to be visible to those who see students every day - families and teachers. We often hear from families that their students are feeling stronger about math in general. We also see kids coming to class early, or returning year after year, which considering that our classes aren't required and happen after school, is a sign that students (and families) believe they are benefitting from the classes.

When surveyed, 100% of parents would recommend our classes to a friend, and 100% said that their children were appropriately challenged. Over 80% of students surveyed stated that their confidence in math has increased as a result of our classes. This is remarkable, given that a sizable fraction of our students reported being maximally confident in math.

[15] The averages and totals do not include Art of Problem Solving students, which make up 10-15 students each year, because these numbers aren't readily available on the dashboard.

Alignment to State Standards

Much like a house that needs to have a strong foundation and all of the floors built up, so does math knowledge. A student who cannot subtract will have difficulty with division, and a student who struggles with order of operations will struggle in solving algebraic equations.

During math class, students focus on grade-level standards, building the "current level". Out of school time, we focus on filling in the levels from before, and putting in supports for future levels, so that when students do get to those grades, their math "buildings" can be stronger. The goal of CMC is to complement school math instruction to have students be as ready as they can for honors and AP high school courses, and, if they choose, STEM majors and careers.

For example, 2-digit and 3-digit addition is a 2nd grade standard, yet many students we have personally worked with in grades 3 and up aren't able to add 17 and 100 (or 17 and 99) in their heads. They end up using column addition every time they need to add (sometimes even for single digits, like 7 + 4) and often make silly mistakes, ending up with the wrong answer. Students may be doing great on the grade-level standards in 3rd grade (multiplication, for example), but not know how to add accurately and efficiently.[16]

On the flip side, students may not be familiar with algebraic concepts or geometric ones that are needed for success in high school and beyond. We do the work of filling in the gaps and strengthening mental math, while also previewing concepts from future years (algebra and geometry fundamentals), while also teaching students self-efficacy and perseverance, by asking them to solve problems that aren't just like what they saw in the textbook or in a previous problem, but require a mental leap.

We are working backwards: we know that not enough students from underrepresented groups are ending up in STEM careers, and high school honors and AP STEM classes. From working at the high school, middle school and elementary school level, and countless conversations with teachers, parents, and thousands of hours of teaching, we work on the skills that are needed for success, as well as the attitudes ("math is fun," "I (the student) can be successful in math," "math is important to know"). We work on preparing students for those classes by giving them "honors" math in elementary and middle school. The exact content depends on a particular child's math needs, but the level of complexity is high, no matter the topic.

This is in contrast to programs such as iReady, which give the same basic content to every child. For example, if a 7th-grader is struggling with addition, they would be given the same content as a child who is learning addition for the first time in 2nd grade. We find that a variety of addition techniques, with thinking required to decide which one to use, works better both for strong students who are younger, and struggling students who are older.

[16] Conversation with a Cambridge math educator.

CMC Measured Success

In measuring the success of our Beast Academy/AoPS program, we have analyzed the following:

1) Number of problems solved by students has been steadily growing, from an average of 263 per student per academic year to over 500 ;
2) The number of students participating in the program has been growing steadily, from 37 to 100, despite the general movement away from online learning, using word of mouth;
3) Students are noticeably growing in math confidence and readiness for STEM careers, as reported by 87% of parents surveyed.

Program Snapshot: Math Circle Classes

Student Profile of the Math Circle Program

Students in grades 1-8 can attend during school (at lunch), after school, or on weekends. We have students who are in one more more of the following categories:

- students who are looking to experience success in math;
- students who love math and are looking for more challenge;
- students whose families are in underrepresented or low-SES groups, who are looking to build confidence and problem-solving skills because they are concerned about the difficulties their children will likely experience if they choose to pursue STEM-intense careers;
- students who want to do something fun with their friends.

We offer many different ways to engage so that the broadest crosssection of students could participate. Over 75% of our students in math circles have been from our target groups.

Goals of the Math Circle Program

Similar to our Beast Academy/AoPS classes, Math Circle classes focus on different aspects of math learning in order to ensure we are raising mathematically competent citizens and diversifying the STEM pipeline.

Outcomes of the Math Circle Program

1) Hundreds of students choose to engage in deep mathematics on their own time each week;

2) Students' ability to reason mathematically, to model situations using math, and to discuss math with others improve;
3) External stakeholders have noted the changes in students as a result of participation in the program: greater interest in math, greater willingness to try unfamiliar, challenging activities;
4) Students are teaching each other and grown-ups.

Alignment to State Standards

By teaching math with a low floor (usually just addition and subtraction for lower grades, and multiplication added for higher grades) and a very high ceiling (college-level problems), we focus on standards for mathematical practice in our math circle classes.

Particular emphasis is paid to the following Massachusetts practice standards:

- Standard 1 - Making sense of problems and persevering in solving them
- Standard 2 - Reasoning abstractly and quantitatively
- Standard 3 - Constructing viable arguments and critiquing the reasoning of others
- Standard 4 - Modeling with mathematics.
- Standard 7 - Looking for and making use of structure and Standard 8 - Looking for and making use of repeated reasoning, are also emphasized.

CMC Measured Success

1) Our program grows mostly by word of mouth. We know our families and partners are happy with the work we are doing since they are bringing their friends and colleagues along.
2) We conduct periodic surveys of students and families, which show us a high degree of efficacy. Examples of survey responses, below:

> *"I liked how fun the math was."*
> **7th-grade boy at Vassal Lane Upper School**
>
> *"Math circle could be longer and happen more days a week. It could be 1 hour and 30 mins long and happen 3 days a week."*
> **4th grade boy, Fletcher-Maynard Academy**
>
> *"It makes math fun, but at the same time I feel like I'm learning."*
> **4th-grade boy, Morse School**
>
> *"I got to learn more things like during the puzzles, we would take turns and share ideas to find out how it can be solved."*
> **5th-grade girl, Graham & Parks School**
>
> *"I liked learning math and hard problems that I got to figure out with my friends."*
> **3rd grade girl, Morse School**

"[What I liked most was] being able to impress parents."
5th-grade boy, Morse School

3) Curriculum CMC has developed 3-4 years ago for 4th-grade students that was appropriately challenging for students just starting with CMC, cannot be reused with 4th grade students who have been at Cambridge Math Circle for a year or more: they simply are too strong as problem-solvers and need a greater challenge. [17]
4) A number of families are engaging in math tasks and games from our math circle classes with their students, leading to rich conversations at home.

Conclusion

As Cambridge Math Circle continues to grow and evolve, we aim to be able to serve more Cambridge children within their own elementary and upper schools, and strive to deepen our partnerships with schools and other local organizations, including the STEAM Initiative. We hope to eventually become an official partner to the Cambridge Public School District. Together with all of the stakeholders, we are working towards a present and a future where every child has a positive self-concept and strong preparation for any math required in college and/or career.

[17] Problems that were very challenging for our earliest classes of community students became almost trivial for students who have worked with us for a year or more, once they reach the same grade level. Problems that used to take 10-15 minutes of active discussion for, say 4th graders to solve, were solved immediately by 4th graders with a year or more of CMC experience, so we had to keep upping the difficulty of the problems. For students joining us for the first time (most of our school math circle students), we take the difficulty to the appropriate level for them, teaching from the same materials, with modifications.